1 Number and algebra

What have you done today? Did you:

- solve a puzzle?
- look at something on the internet?
- take some medication?
- use your cell phone?

What might you be doing in ten years' time? Do you think that you will be:

- working in a design office?
- using spreadsheets to analyse large amounts of data?
- involved in making financial or investment decisions?
- looking into the global logistics of moving goods and materials?

In all of the above activities, algorithms are used to calculate the most efficient way of solving a problem. An **algorithm** can be defined as *a detailed set of instructions that leads to a predictable result or a set of instructions that helps you to finish a task as efficiently as possible.*

The mathematical skills of manipulating numbers and using algebra provide a powerful tool in devising, understanding and applying algorithms.

Prior learning topics

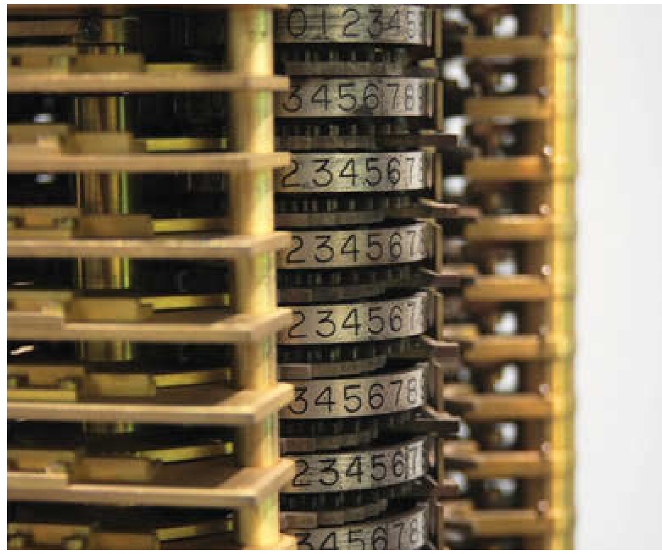
It will be easier to study this topic if you:

- are comfortable with the four operations of arithmetic (addition, subtraction, multiplication, division)
- understand the **BIDMAS** order of operations (**B**rackets, **I**ndices, **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction)
- can use integers, decimals and fractions in calculations
- can recognise prime numbers, factors and multiples
- know some simple applications of ratio, percentage and proportion
- can identify intervals on the real number line
- are able to evaluate simple algebraic expressions by substitution
- are comfortable with basic manipulations of algebraic expressions such as factorisation, expansion and rearranging formulae
- understand how to use the inequalities $<, \leq, >, \geq$
- are familiar with commonly accepted world currencies such as the euro, United States dollar and Japanese yen.

Chapter 1 Number

In this chapter you will learn:

- about the different types of numbers
- how to make approximations
- about estimating and how to check solutions using your GDC
- how to express very small and very large numbers and perform operations with them
- about SI (Système International) units.



An Analytical Engine

Ada Lovelace had a very clear idea of the role of the analytical engine, saying: 'The Analytical Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform. It can follow analysis; but it has no power of anticipating any analytical revelations or truths. Its province is to assist us in making available what we are already acquainted with.' This is 'Note G' from Ada Lovelace's translation and commentary of a 1842 document called *Sketch of the Analytical Engine* by L.F. Menabrea (information sourced from www.allonrobots.com/ada-lovelace.html). This is a good description of all computers and calculating machines.

Charles Babbage (1791–1871) is described in some histories as 'the father of computing'. A mathematician and inventor, he was looking for a method to improve the accuracy of mathematical tables. These lists of numbers included squares and square roots, logarithms and trigonometric ratios, and were used by engineers, navigators and anyone who needed to perform complex arithmetical calculations. The tables were notorious for their inaccuracy, so Babbage designed a machine to re-calculate the numbers mechanically. In 1822, the Royal Society approved his design, and the first 'difference engine' was built at the inventor's home in London.

Babbage went on to develop an improved machine, called an 'Analytical Engine', which is now seen as the first step towards modern computers. He worked on this machine with Ada Lovelace (1815–1852), who invented the punched cards that were used to 'programme' the machine. Lovelace is considered to be the first computer programmer.

1.1 Different types of numbers

The natural numbers, $\mathbb{N}$

The **natural numbers** ($\mathbb{N}$) are the counting numbers, the first numbers that people learn and use.

1, 2, 3, 4, 5, … are all counting numbers.

−1, −2, −3, … are negative numbers and therefore are **not** natural numbers.

1.5, 2.3, 6.7 are not whole numbers and therefore are **not** natural numbers.

When small children learn to count, they soon realise that '3' will always mean the same quantity, three, whether they are counting apples, sheep or chairs.

Natural numbers are also the numbers typically used for comparison. For example, you might say, 'I have read all seven Harry Potter books; the second and the seventh were my favourites.'

Natural numbers can be shown on a **number line**:

We can also write the natural numbers as follows:

$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$, where $\mathbb{N}$ is the symbol for natural numbers.

The curly brackets { } enclose all the numbers represented by the symbol $\mathbb{N}$. These brackets signify that the natural numbers form a **set** (or collection).

The place of zero among the natural numbers is a point of debate. Not all definitions of the natural numbers include zero. In this IB course, the natural numbers do include the number zero.

FF *Sets and set notation are explained in Chapter 8.*

The integers, $\mathbb{Z}$

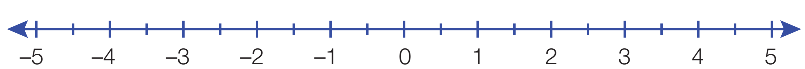
Natural numbers can only be used for counting in one direction: left to right along the number line, starting from zero. It is often useful to be able to count down to below zero. This is where a larger set of numbers, called the 'integers', comes in.

The **integers** are defined as all whole numbers: positive, negative and zero.

–79, –2 and 1 0001 are all integers.

–9.99, $1\frac{1}{4}$ and 1 0001.4 have decimal or fractional parts and are therefore **not** integers.

Integers can also be shown on a number line:



The number line is infinitely long and extends from negative infinity to positive infinity. The concept of **infinity** is very important to mathematicians, but it is difficult to define. Imagine a line, divide it in half, then divide in half again, and continue in this way. No matter how many times you divide the line in half, you can always divide it again. So you can do an 'infinite' number of divisions. The symbol for infinity is ∞.

We can use set notation to write the integers, as we did for natural numbers:

$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$, where $\mathbb{Z}$ is the symbol for integers.

$\mathbb{Z}^+$ is defined as the set of all positive whole numbers, or integers greater than zero.

$\mathbb{Z}^-$ is defined as the set of all negative whole numbers, or integers less than zero.

The symbol $\mathbb{Z}$ comes from 'Zahlen', the German word for numbers.

The rational numbers, $\mathbb{Q}$

A **rational number** is a number that results when one integer is divided by another. Dividing one integer by another creates a **ratio**, which is where the term '**ratio**nal numbers' comes from.

If q is a rational number, then $q = \frac{a}{b}$, where a and b are both integers, with $b \neq 0$ (a is called the **numerator** and b is called the **denominator**).

$-\frac{7}{2}, \frac{1}{5}, \frac{28}{9}, 5\frac{3}{4}$ are all rational numbers.

Note that mixed numbers such as $5\frac{3}{4}$ are also rational numbers. This is true because you can rewrite them as **improper fractions**, e.g. $5\frac{3}{4} = \frac{23}{4}$.

Rational numbers are often written as decimals, which can make it less obvious that they are rational numbers. For example:

−3.5, 0.2, 3.111111111…are all rational numbers.

You can check by writing each as a fraction:

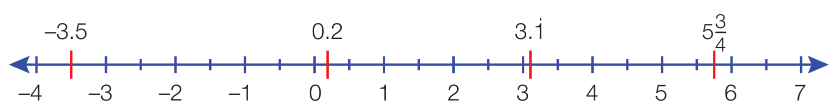
$-3.5 = -\frac{7}{2}$, so −3.5 is a rational number.

$0.2 = \frac{2}{10} = \frac{1}{5}$, so 0.2 is a rational number.

$3.111111111\ldots = \frac{28}{9}$, so 3.111111111… is a rational number.

Numbers like 3.111111111…, where a digit (or group of digits) repeats forever, are called 'recurring decimals' and are often written with a dot above the number that repeats, e.g. $3.\dot{1}$. All recurring decimals are rational numbers.

Rational numbers can also be shown on a number line:



hint

$\mathbb{Q}$ stands for **quotient**. A quotient is the result of a division.

hint

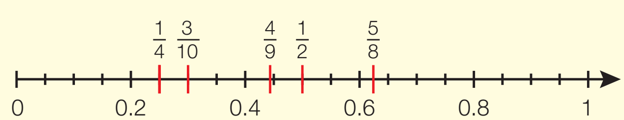
Remember that a mixed number is made up of a whole number and a fraction, e.g. $1\frac{2}{3}$. An improper fraction is a fraction where the numerator is larger than the denominator, e.g. $\frac{5}{4}$.

hint

All integers are rational numbers too, e.g. $5 = \frac{5}{1}$ is a fraction with denominator 1.

Worked example 1.1

Q. Find a rational number between $\frac{1}{4}$ and $\frac{5}{8}$.



A. Possible fractions include $\frac{3}{10}, \frac{4}{9}, \frac{1}{2}$

There are an infinite number of fractions to choose from. A number line can help you make a correct choice.

The irrational numbers

An **irrational number** is a number that **cannot** be written as a fraction. The decimal part of an irrational number has no limit to the number of digits it contains and does not show a repeating pattern.

A well-known example of an irrational number is π (pi). You know this as the ratio of the circumference of a circle to its diameter: $\pi = 3.14159265\ldots$

Modern computers enable mathematicians to calculate the value of π to many millions of digits, and no repeats of groups of digits have been found!

Other irrational numbers include ϕ (the 'golden ratio') and the square roots of **prime numbers**:

$$\phi = 1.6180339887498948482\ldots$$

$$\sqrt{2} = 1.414213562373\ldots$$

$$\sqrt{7} = 2.6457513110645\ldots$$

Irrational numbers do not have their own symbol; however, they are sometimes represented by $\bar{\mathbb{Q}}$. The bar above the $\mathbb{Q}$ indicates that the irrational numbers form the **complement** (opposite) of $\mathbb{Q}$ (the rational numbers). Why do you think irrational numbers have not been given their own symbol?

The real numbers, $\mathbb{R}$

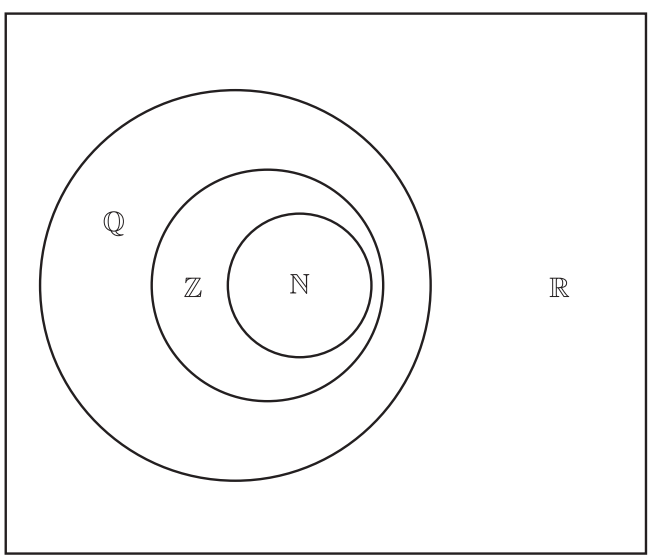
The **real numbers** are all the numbers that can be represented on a number line.

They include the natural numbers, integers, rational numbers and irrational numbers.

It may be easier to think of the various types of real numbers in the form of a **Venn diagram**.

A Venn diagram uses shapes (usually circles) to illustrate mathematical ideas. Circles may overlap or lie inside each other. In the example below, $\mathbb{N}$ is inside $\mathbb{Z}$ because all natural numbers are integers.

FF *You will learn about Venn diagrams in Chapter 8.*



It can be proved that there are an infinite number of real numbers. Georg Cantor constructed the proof in 1874. It is a powerful proof, surprisingly simple, and worth reading about and understanding. The ideas behind the proof are very profound and have had an important influence on the work of mathematicians following Cantor.

exam tip

In this course, all the numbers you will encounter are real numbers.

Irrational numbers are included in the definition of real numbers. In the Venn diagram they are represented by the region **outside** the natural numbers, integers and rational numbers.

The history of numbers follows the development of human history. The modern system of numbers is based on Arabic notation, and also uses ideas from Indian mathematics, such as zero.

Worked example 1.2

Q. Mark each cell to indicate which number set(s) the number belongs to.

	-2	$\frac{3}{7}$	$\sqrt{13}$	3π	10 000
Irrational					
$\mathbb{N}$					
$\mathbb{Z}$					
$\mathbb{Q}$					
$\mathbb{R}$					

A.

	-2	$\frac{3}{7}$	$\sqrt{13}$	3π	10 000
Irrational			×	×	
$\mathbb{N}$					×
$\mathbb{Z}$	×				×
$\mathbb{Q}$	×	×			×
$\mathbb{R}$	×	×	×	×	×

The cells have been filled in to show that:

- -2 is an integer, a rational number and a real number
- $\frac{3}{7}$ is a rational number and a real number
- $\sqrt{13}$ is an irrational number and a real number
- 3π is an irrational number and a real number
- 10 000 is a natural number, an integer, a rational number and a real number.

Do you think that numbers have always been in existence, waiting to be discovered, or are they an invention of mathematicians?

Worked example 1.3

Q. Look at the list of numbers:

$\sqrt{5}$, $-\frac{3}{7}$, π, -5, 7, 2^3.

(a) Which numbers are integers?

(b) Which numbers are both rational and negative?

(c) Which numbers are not rational?

(d) Which numbers are not natural?

A. (a) $-5, 7, 2^3$

These are all integers; note that $2^3 = 8$.

(b) $-\frac{3}{7}$ and -5

Remember that -5 is also a rational number $\left(-\frac{5}{1}\right)$.

continued . . .

$\sqrt{5}$ cannot be written as a fraction, and as a decimal it does not have a limit or a recurring pattern; the same is true for π.

(c) $\sqrt{5}$ and π

These are not counting numbers, so they are not natural.

(d) $\sqrt{5}, -\frac{3}{7}, \pi, -5$

Exercise 1.1

1. Look at each of the following statements and decide whether it is true or false. If it is false, give the correct statement.

(a) 2.4 is a rational number.

(b) $6 + (-2)$ gives an answer that is a natural number.

(c) $\sqrt{17}$ is a rational number.

(d) 1.51 is an irrational number.

(e) 5π is a real number.

(f) An irrational number is never a real number.

(g) If you add two integers, the answer will not be an integer.

(h) If you divide one integer by another, the answer is a real number.

Srinivasa Ramanujan was a mathematical genius. Born in 1887 near Madras in India, he was fascinated by numbers and made many significant contributions to the theory of numbers. He is also well known for his collaboration with the famous mathematician G. H. Hardy of the University of Cambridge.

2. Write down a number that is:

(a) a real number and an integer

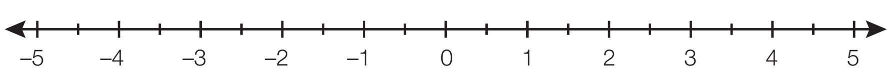
(b) a rational number, but not an integer

(c) a real number and an irrational number

(d) a natural number that is also rational.

3. Copy the number line below, and put these numbers in the correct place on the number line:

$$\frac{12}{13}, -\sqrt{3}, 3.1, -1 + \pi, -4.2, -4.25, \sqrt{13}$$



4. (a) Put the following numbers in ascending order:

$$12, -5.\dot{1}, \sqrt{3}, \sqrt{2}, -2, 0, \frac{6}{7}, -2.5$$

(b) Write down the natural numbers in the list.

(c) Write down the integers in the list.

(d) Write down the rational numbers in the list.

(e) Which numbers have you not written down? Why?

5. Look at each of the following statements, and use the given words to fill in the blanks, making a correct sentence.

rational, irrational, real, negative, natural, integer

(a) If you add two natural numbers, the answer is a(n) .... number.

(b) If you add two rational numbers, the answer can be a(n) .... number, a(n) .... (number) or a(n) .... number

(c) If you add a negative integer to another negative integer, the answer is a(n) ....

(d) If you add a natural number to an irrational number, the answer is a(n) .... number.

(e) If you add a natural number that is greater than 12 to an integer between –10 and zero, the answer is a(n) ....

6. (a) Find a rational number between $\frac{4}{5}$ and $1\frac{3}{4}$.

(b) Find a rational number between $2\frac{2}{3}$ and $3\frac{3}{4}$.

7. Look at the list of numbers below, and use it to answer the following questions:

43, 2, 21, 15, –6, 17, 6, –4, 13

(a) Write down the prime numbers.

(b) Write down the multiples of three.

(c) Write down the even numbers.

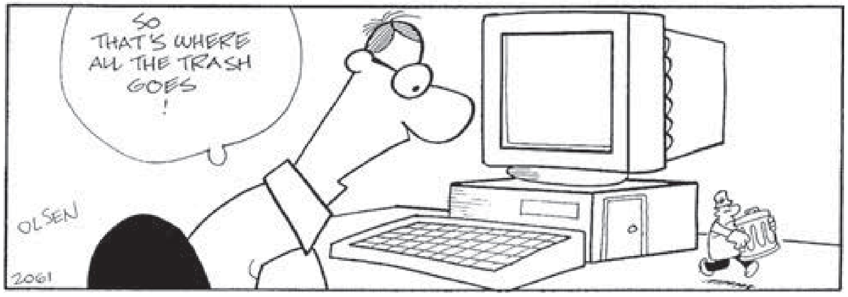
(d) Which number have you written for both (a) and (c)?

1.2 Approximation and estimation

Modern calculators and computers allow us to calculate with great precision. However, in everyday life most people use estimates and few people are comfortable with very large, very small, or very long numbers.

With the increasing use of calculators and computers, it has become more important to know how to estimate a rough answer and to 'round' the results obtained by technology so that we can use them sensibly.

There is a well-known saying about modern technologies: 'Garbage **in**, garbage **out**'. This means that you need to put sensible input into your calculator or computer, in order to be able to make sense of what you get as an output.



We also need to agree upon our methods of **rounding**; for example, if one shopkeeper always rounds her prices **down** to the nearest cent, while her rival always rounds his prices **up** to the nearest cent, what will be the result?

Your graphical display calculator (GDC) is a piece of modern technology that is fundamental to this course. You cannot achieve a good understanding of the material without it! It is important that you are able to:

- estimate a rough answer before using your GDC
- put information into the GDC accurately
- sensibly use the GDC's ability to work to nine decimal places.

It is very easy to accidentally press the wrong key on your GDC, which could lead to an incorrect answer. If you already have an estimate of what you think the answer will be, this can help you to identify if the answer your GDC gives you looks about right; if it is completely different to what you expected, perhaps you pressed a wrong key and need to enter the calculation again.

Approximation by rounding

Rounding is an idea that many people apply instinctively.

When someone asks you 'how long was that phone call?', you would usually not reply 'nine minutes and thirty-six seconds'. You would probably say 'about ten minutes', rounding your answer to the nearest minute. The 'nearest minute' is a **degree of accuracy**.

When asked to round a number, you will normally be given the degree of accuracy that you need. Some examples are:

- to the nearest yen
- to the nearest 10 cm
- to the nearest millimetre
- to the nearest hundred
- to one decimal place
- to three significant figures.

To round numbers you can use either a number line or the following rule:

- If the digit to the right of the digit you are rounding is less than five (<5), then the digit being rounded stays the same.
- If the digit to the right of the digit you are rounding is five or more (≥ 5), then the digit being rounded increases by one.

hint

Remember the meaning of the following symbols:
$<$ 'less than'
$\geq$ 'greater than or equal to'
$\approx$ 'approximately equal to'

Worked example 1.4

Q. Use the rule above to round the following numbers:

(a) 1056.68 yen to the nearest yen

(b) 546.21 cm to the nearest 10 cm

(c) 23.35 mm to the nearest mm

(d) 621 317 to the nearest 100.

A. (a) 1056|.68 yen ≈ 1057 yen

Look at the last digit before the decimal point, 6. The digit to its right is also 6. As 6 > 5, we round the digit before the decimal point to 7.

(b) 54|6.21 ≈ 550 cm

The digit we are rounding is 4, which is followed by 6. As 6 > 5 we round 4 up to 5.

(c) 23|.35 mm ≈ 23 mm

The digit we are rounding is 3, and the digit following it (the first digit after the decimal point) is also 3. As 3 < 5 we leave the digit we are rounding unchanged.

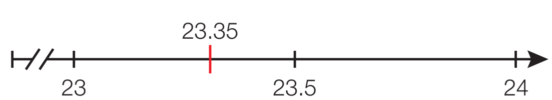
(d) 621 3|17 ≈ 621 300

The digit we are rounding is 3, and it is followed by 1. As 1 < 5 the digit we are rounding is unchanged.

hint

In part (d) of Worked example 1.4, you need to replace the last two digits '1' and '7' by zeros in order to keep the number the correct size. The 3 represents 300, so 317 to the nearest 100 is 300, and therefore '1' and '7' must be replaced by '00'.

You may find it easier to visualise the rounding process using a number line. For example for part (c) above:



23.35 is closer to 23 than to 24, so 23.35 mm = 23 mm to the nearest mm.

Learning links

1A Place order structure of numbers

You might find it useful to remember the 'place order' (or 'place value') of numbers when considering degrees of accuracy. It is also useful when making sure the rounded value is the same order of magnitude (size) as the original number (millions must remain millions and thousands must remain thousands, etc.).

1	0	0	0	0	0	0	.	0	0	0
millions	hundreds of thousands	tens of thousands	thousands	hundreds	tens	units		tenths	hundredths	thousandths

Exercise 1.2

1. (a) Tom is 168.5 cm tall. Give his height to the nearest centimetre.

(b) A film lasts for one hour and forty-seven minutes. Round this time to the nearest five minutes.

(c) There were 3241 people at a hockey match. Give the number to the nearest hundred.

(d) A camera costs 219 AUD. Round this price to the nearest 10 dollars.

(e) The area of a field is 627.5 m^2. Give this area to the nearest square metre.

(f) A sheep weighs 75.45 kg. Give the mass to the nearest kilogram.

2. Round these numbers to the nearest 10.

(a) 49 (b) 204 (c) 319

(d) 2153 (e) 20 456

3. Round these numbers to the nearest 100.

(a) 346 (b) 2011 (c) 85

(d) 67 863 (e) 708 452

4. Calculate the following values using your GDC. Give your answer to the nearest integer.

(a) $\sqrt{4.56}$ (b) 3.17^3 (c) $15 \sin 60°$ (d) $\pi \times 6^2$

exam tip

Some exam questions may ask you to give your answer to a specified number of decimal places, so you need to know how to do this. **However**, remember that if a specific degree of accuracy has **not** been requested, you **must** give the answer to three significant figures (see the section on 'Significant figures' below).

Decimal places

Standard calculators can give you answers to several **decimal places (d.p.)**. A GDC can give you answers to as many as nine decimal places!

For most practical purposes, nine decimal places is far too many figures, so you need to be able to round to a given number of decimal places.

To round a decimal number, you can use the decimal point as your first reference, and then use the number line, the '< or ≥ 5' rule, or your GDC to round up or down appropriately.

Worked example 1.5

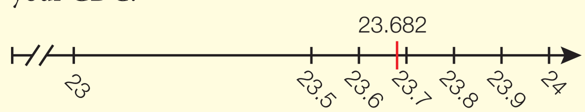
Q. Round 23.682 to 1 decimal place using:

(a) a number line

(b) the < or ≥ 5 rule

(c) your GDC.

A. (a)



Draw your number line with intervals of one decimal place and see which number 23.682 is closest to.

23.682 is closer to 23.7 than to 23.6

23.682 = 23.7 (to 1 d.p.)

(b) 23.6|82

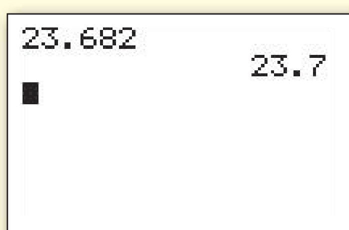
8 > 5

so 23.682 = 23.7 (to 1 d.p.)

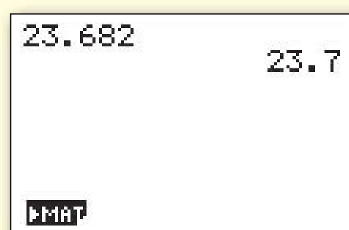
6 is in the 1 d.p. position; 8 is the digit to the right of 6 and is greater than 5, so round 6 up to 7.

(c)

TEXAS



CASIO



So 23.682 = 23.7 (1 d.p.)

See '*1.1 Rounding*' on page 648 of the GDC chapter if you need a reminder of how to round using your GDC.

Can you use equals signs in these approximations? Or should you always use the 'approximately equal to' sign ≈ ?

When using your calculator, be practical: it is better use all the digits it provides in your calculations to get an accurate answer, but you do **not** need to write down every figure that your GDC gives you in each step of your working. Instead, store the long numbers in the memory of your GDC so they can be used in future calculations, but only **write down** a rounded value in your working (see '*22.2D Using the GDC memory*' on page 643 of the GDC chapter). Make sure you state the degree of accuracy that you have used when writing a rounded answer. It is fine to use abbreviations such as d.p. and s.f.

Exercise 1.3

1. Write the following numbers to
(i) 1 decimal place; (ii) 2 decimal places.

(a) 6.8152 (b) 153.8091 (c) 17.966 (d) 0.1592

2. Use your GDC to calculate the following.
Give your answers to one decimal place.

(a) $76.95 \times 2.1 \div 3.86$ (b) $(3.8 + 2.95)^2$ (c) $\sqrt{3^2 + 4^2 + 5^2}$

3. Use your GDC to calculate the following.
Give your answers to two decimal places.

(a) If $r = 6.8\,\text{cm}$, then $\pi r^2 =$

(b) If $r = 3.2\,\text{cm}$ and $h = 2.9\,\text{cm}$, then $\pi r^2 h =$

(c) If $r = 4.2\,\text{cm}$ and $h = 3.95\,\text{cm}$, then $\frac{1}{3}\pi r^2 h =$

Significant figures

On the instruction page of each examination paper for the IB Mathematical Studies SL course (the page on which you write your name), students are asked to give all final answers to an exact value, or to **three significant figures** if the question does not request a specific degree of accuracy.

exam tip

You should practise rounding answers to three significant figures from the beginning of the course, so that you will be confident when you take the final examinations.

Rounding to a certain number of **significant figures (s.f.)** is the most flexible system of rounding, as you can use it for numbers of any size as well as for numbers that have no decimal point. This means you can use it for very small numbers as well as for very large ones.

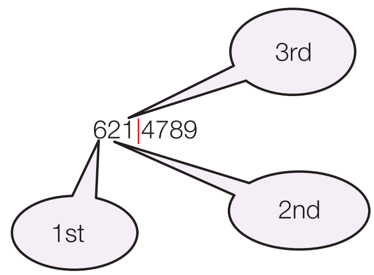
First we need to understand what is meant by 'significant figure':

- In a number that is greater than 1 (e.g. 143) the **first significant figure** is the **first (leftmost) digit** in the number, the **second significant figure** is the **second** digit from the left, and so on. So, in the number 143, '1' is the first significant figure, '4' is the second significant figure, and '3' is the third significant figure.
- In a number that is less than 1 (e.g. 0.0415), the first significant figure is the **first non-zero digit after the decimal point**, the second significant figure is the next digit to the right of the first significant figure, and so on. So, in the number 0.0415, '4' is the first significant figure, '1' is the second significant figure, and '5' is the third significant figure.
- In a number with figures on both sides of the decimal point (e.g. 78.2), the first significant figure is the first (leftmost) digit in the number, the second significant figure is the second digit from the left, and so on. So, in the number '78.2', '7' is the first significant figure, '8' is the second significant figure, and '2' is the third significant figure.

To round to **three significant figures**:

1. Find the third significant figure in the original number.
2. Look at the digit to the right and round the third significant figure according to the '< 5 or ≥ 5' rule.
3. Check the rounded number is the correct magnitude. If you need to, replace digits on the right of the third significant figure with zeros. Be careful to keep track of the number of digits that need to be replaced: the rounded number must be of the same magnitude as the original number (millions must remain millions, hundreds stay as hundreds, etc.)

For example, to round 6 214 789 to three significant figures:



1 is the third significant figure and the next digit to the right is 4. As 4 < 5, you do not change the third significant figure.

Replace the '4', '7', '8' and '9' to the right of the third significant figure with zeros.

So 6 214 789 = 6 210 000 (3 s.f).

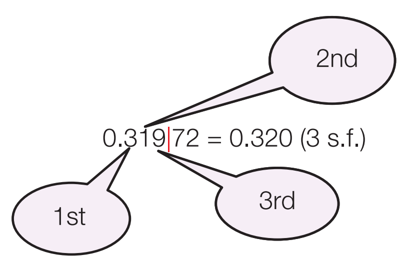
hint

In Worked example 1.4 the answers were written using the symbol ≈ which means 'approximately equal to'. In this case no degree of accuracy was included with the answer so this symbol was used to indicate that the answer was rounded. So, 1056.68 yen ≈ 1057 yen indicates that 1057 is 'approximately equal to' and therefore states it is a rounded value. If the degree of accuracy was included, we would have used the = sign: 1056.68 = 1057 yen (to the nearest yen).

Notice that the answer is written as 6 214 789 = 6 210 000 **(3 s.f.)**; it includes the comment '(3 s.f.)' at the end. This comment means 'to three significant figures' and indicates that the values are 'equal' *at that degree of accuracy*. The addition of the comment '(3 s.f.)' is required because 6 214 789 is not *exactly* equal to 6 210 000, so without it, the statement '6 214 789 = 6 210 000' would not actually be true. Some texts prefer to use the notation 6 214 789 → 6 210 000 (3 s.f.).

In some numbers, you will find that rounding the significant figure up means going from '9' to '10', which means you need to change both the significant figure and the number **before** it.

For example, to round 0.31972 to three significant figures:



The third significant figure is 9. Because the next digit, 7, is greater than 5, 9 is rounded up to 10. In this case, you need to carry the '1' from '10' over to the second significant figure (making it '2'), and you *must* keep a zero (the '0' from '10') in the position of the third significant figure, to maintain the degree of accuracy you were asked for. An answer of 0.32 would be incorrect, because it contains only **two** significant figures.

Worked example 1.6

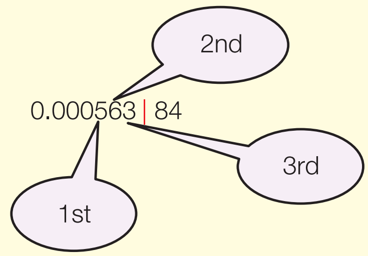
Q. Round the following numbers to three significant figures.

(a) 0.00056384

(b) 4.607054

A. (a)

Find the third significant figure and look at the digit after it; as 8 > 5, round 3 up to 4.

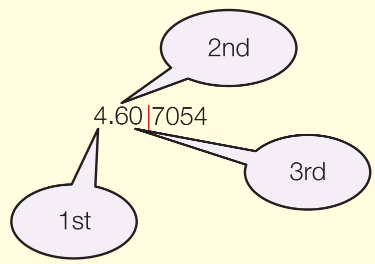


Remember to retain the zeros at the beginning of the number as they give the correct position for the later digits.

0.00056384 = 0.000564 (3 s.f.)

(b)

The third significant figure is '0'; as the next digit to the right is 7 and 7 > 5, round 0 up to 1.



The '7', '0', '5' and '4' after the third significant figure are dropped.

4.607054 = 4.61 (3 s.f.)

Exercise 1.4

1. Write the following numbers to three significant figures.

(a) 93.5159 (b) 108.011 (c) 0.0078388

(d) 8.55581 (e) 0.062633

2. Use your GDC to calculate the following. Give your answers to 3 s.f.

(a) $6.96 \times 2.15 \div 4.86$ (b) $(8.3 + 1.95)^3$

(c) $\sqrt{6^2 + 5^2 + 4^2}$ (d) $\sqrt{6} + \sqrt{5} + \sqrt{4}$

3. Calculate the following, giving your answers to 3 s.f.

   (a) The area of a rectangle with a length of 6.9 cm and a width of 6.3 cm.

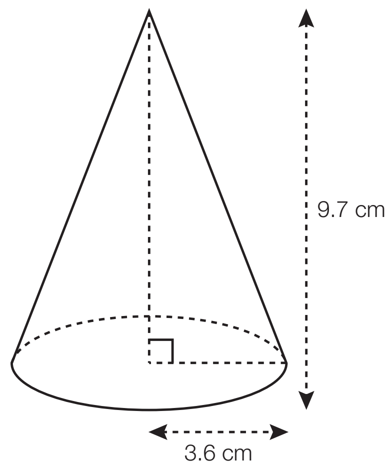
   (b) Use the formula $C = 2\pi r$ to calculate the radius of a circle with a circumference of 20 cm.

   (c) Use Pythagoras' theorem $a^2 + b^2 = c^2$ to calculate the length of the hypotenuse of a right-angled triangle whose shorter sides measure 7 cm and 5 cm.

4. Find the volume of this cone.
   Give your answer to three significant figures.

hint

You can look up the formula for the volume of a cone in the Formula booklet. $a = \pi r^2$



Important considerations for rounding

You can round numbers during a calculation, but be aware that if you round a number too early, you could significantly change the final answer.

The example below demonstrates this by calculating the volume of a cylinder using a rounded value for the area of its circular base, and comparing this to the volume obtained from using the exact area.

Worked example 1.7

Q. (a) Find the area of a circle with radius of 6.81 cm.

(b) Write down your answer from (a) to three significant figures.

(c) Use your answer from (b) to calculate the volume of a cylinder with base radius 6.81 cm and height 14.25 cm using the formula $V = \text{base area} \times \text{height}$.

continued . . .

(d) Re-calculate the volume of the cylinder using the formula $V = \pi r^2 h$

(e) Find the difference between the answer to (c) and the answer to (d).

You can find the formula for the area of a circle in the Formula booklet.

A. (a) Area of circle:

$A = \pi r^2$

$= \pi \times 6.81^2 = 145.69\ldots \text{cm}^2$

In 145.69…, '5' is the third significant figure; as 6 > 5, round 5 up to 6.

(b) $146\,\text{cm}^2$ (3 s.f.)

(c) $V = \text{base} \times \text{height}$

$= 146 \times 14.25 = 2080.5\,\text{cm}^3$

(d) $V = \pi r^2 h$

$= \pi \times 6.81^2 \times 14.25$

$= 2076.15$ (2 d.p.)

(e) $2080.5 - 2076.15 = 4.35\,\text{cm}^3$ (3 s.f.)

Worked example 1.7 helps to demonstrate the importance of rounding at an appropriate stage during a calculation. In part (d) no rounded values were used during the calculation; only the final value was rounded. But in part (c) a rounded value was used within the calculation. The answer from (c) is larger than that in (d); part (e) shows you the difference is $4.35\,\text{cm}^3$. Rounding during a calculation, rather than just at the end of the calculation, can lead to a less accurate estimate; in Worked example 1.7, rounding too early in the calculation led to an overestimate of the cylinder's volume.

If you had $2000\,\text{cm}^3$ of fluid, then either method (part (c) or (d)) would be sufficient for checking whether the bottle is big enough to hold the fluid: both answers are larger than the required $2000\,\text{cm}^3$.

You need to decide how accurate you need the answer to be in order to decide if it is acceptable to round during a calculation.

In the drinks bottle example, the numbers for the volume are sufficiently close that the consequences of the value being wrong are not that important. Can you think of situations where rounding an answer too early could have serious consequences?

Estimation

Using a computer or a calculator gives you a very accurate answer to a problem. However, it is still important to be able to **estimate** the answer that you are expecting. This will help you to realise when you might have made a mistake, and to notice when other people make them too!

For example, Ed is shopping for his class barbecue. He buys 12 cartons of orange juice and 6 bottles of cola. The orange juice costs \$1.42 a carton and the cola costs \$1.21 a bottle.

He estimates the bill as follows:

$$12 \times 1.42 \approx 12 \times 1.5$$
$$= (12 \times 1) + (12 \times 0.5)$$
$$= 12 + 6$$
$$= 18$$
$$6 \times 1.21 \approx 6 \times 1$$
$$= 6$$

Total = \$24

At the cash register Ed is charged \$177.66. This seems far too large a sum, and does not agree with Ed's estimate. What has gone wrong?

There is a record on the cash register of what was entered by the cashier:

$$12 \times 14.2 = 170.4$$
$$6 \times 1.21 = 7.26$$
$$\text{Total} = \$177.66$$

The cashier has put the decimal point for the price of the orange juice in the wrong place. The bill should be:

$$12 \times 1.42 = 17.04$$
$$6 \times 1.21 = 7.26$$
$$\text{Total} = \$24.30$$

If Ed had not made the estimate before he paid, he might not have questioned the amount he was charged and could have paid way too much!

To estimate quickly:

1. Round the numbers to simple values (choose values that will allow you to do the calculation using mental arithmetic).
2. Do the calculation with the simplified numbers.
3. Think about whether your estimate seems too big or too small.

Worked example 1.8

Q. Zita's mother has told her that she can redecorate her room. The room measures 5.68 m by 3.21 m. Zita wants to buy new carpet for the floor.

(a) Estimate the floor area of Zita's room to the nearest metre.

(b) Based on your answer from (a), what area of carpet do you suggest Zita should buy?

(c) Find the accurate floor area using your GDC and determine whether your suggestion in part (b) is sensible.

continued . . .

Use the '< or ≥ 5' rule to round each value in the calculation to the nearest metre.

A. (a) $5.68 \times 3.21 \approx 6 \times 3 = 18\,\text{m}^2$

One value was rounded up and the other was rounded down, so the result could be either too high or too low.

(b) To be sure that Zita buys enough carpet, it is best to assume that the estimate is too low. She probably needs to buy $19\,\text{m}^2$.

(c) Accurate calculation:

$5.68 \times 3.21 = 18.23\,\text{m}^2$ (2 d.p.)

so $18\,\text{m}^2$ would not have been enough; the suggestion to buy $19\,\text{m}^2$ was correct.

Exercise 1.5

1. Round each figure in these calculations to one significant figure, and use the rounded values to estimate an answer to the calculation. Check your estimate by using your GDC.

(a) $5.8^2 \times 2.25$ (b) $11.7 \times 2\pi$ (c) $37.9 \div 5.16$

2. Nike measures the base of a parallelogram as 14.7 cm and its height as 9.4 cm.

(a) Calculate the area of the parallelogram:

(i) exactly (ii) to 1 d.p. (iii) to 3 s.f.

(b) Nike estimates the answer as $15 \times 9 = 135\,\text{cm}^2$. Calculate the difference between the exact answer and her estimate.

hint

The formula for the area of a parallelogram is in the Formula booklet.

$a = \pi r^2$

3. (a) Write down the following figures to 1 s.f.:

$m = 28.07$ $n = 13.23$ $p = 112.67$

(b) If $q = \dfrac{25m}{n + p}$, use your answers from (a) to give an estimate of the value of q.

(c) Use the accurate values in (a) to find an accurate value for q. Write down all the figures from your calculator display.

(d) Calculate the difference between your estimate in (b) and the accurate answer in (c).

4. Winston estimated the area of a rectangular field as 12 000 m^2. The exact measurements of the field were 383 m × 288 m.

   (a) Calculate the area of the field using the exact measurements.

   (b) What mistake do you think Winston made in his calculation?

Percentage errors

Estimating and rounding values involves error. It is important to establish a way of assessing the magnitude (size) of that error.

Sam estimates the number of people watching a football match. He thinks there are about 14 500. The gate receipts show that there were actually 14 861 people watching the match.

Deepa estimates that there were 380 people attending a school concert. The actual number was 349.

Who gave the better estimate?

14861 − 14500 = 361, so Sam's estimate was too low by 361 people.

380 − 349 = 31, so Deepa's estimate was too high by 31 people.

First impressions suggest that Deepa's estimate is better because the difference between the estimate and the actual value is smaller. But the sizes of the crowds were very different, so just looking at the differences does not give a fair comparison. To compare the results more fairly, we can use **percentage error**. The percentage error works out the difference between the estimate and the actual value **in relation to** the size of the actual value. The smaller the percentage error, the more accurate the estimate is.

$a = \pi r^2$ Percentage error $\varepsilon = \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$, where v_A = approximate (estimated) value and v_E = exact value.

How important is accuracy? Is it more important in one context than in another? Is it more important to be accurate in physics, medicine or finance? Is there a context in which accuracy is not important?

The vertical bars around the fraction mean that we consider only positive values; for instance, if the difference between the exact and approximate values turns out negative, we just drop the minus sign.

Sam's percentage error is $\frac{14861 - 14500}{14861} \times 100 = 2.43\%$

Deepa's percentage error is $\frac{380 - 349}{349} \times 100 = 8.88\%$

So actually, Sam's estimate was the better one because the percentage error is smaller.

Error can also occur in the process of measuring. A physical measurement will always be a rounded value, because we do not have tools that are sensitive enough to measure continuous data to complete accuracy.

Worked example 1.9

Q. At the beginning of a problem, Ben writes $\sqrt{13} = 3.6$.

He chooses to use the rounded answer of '3.6' rather than the exact value of $\sqrt{13}$ in the calculations that follow. So, instead of calculating $(2+\sqrt{13})^3$, he calculates $(2 + 3.6)^3$.

What is his percentage error?

Substitute the exact value and the rounded value into the formula for percentage error.

A. $V_E = (2+\sqrt{13})^3$

$V_A = (2+3.6)^3$

This percentage error is a result of Ben rounding the numbers during the calculations.

$$\text{Percentage error} = \frac{(2+\sqrt{13})^3 - (2+3.6)^3}{(2+\sqrt{13})^3}$$

$$= 0.297\%$$

Exercise 1.6

1. In each of these questions, calculate the percentage error. Give your answer to 3 s.f.
   (a) Anu estimates the length of a piece of rope to be 5.5 m. The accurate length is 5.65 m.
   (b) Miki types 3.96 into his calculator instead of 3.69.
   (c) Ali estimates the crowd at a basketball match as 4500. The true number is 4241.
   (d) Amy tells her mother that the call on her cell phone took only ten minutes. The call really took 14 minutes and 40 seconds.
2. Maria measures the diameter of a cylinder as 8.75 cm and its height as 5.9 cm.
   (a) Using the formula $V = \pi r^2 h$, calculate the volume of the cylinder to two decimal places.
   (b) More accurate measurements give the volume of the cylinder as 342.72 cm^3. Calculate the percentage error between Maria's answer and the accurate value.

1.3 Expressing very large and very small numbers in standard form

Very large and very small numbers are used in many fields of science and economics: for example, earth sciences, astronomy, nanotechnology, medical research and finance. Chemists measure the diameter of an atom, while astronomers measure distances between planets and stars.

Scientists, doctors, engineers, economists and many other people often use very large or very small numbers. For example:

- 'The galaxy of Andromeda is 2 000 000 light years from our galaxy.'
- 'A light year is approximately 9 460 000 000 000 km.'
- 'It is estimated that the volume of petrol reserves worldwide is 177 000 000 000 cubic metres.'
- 'The thickness of this strand of glass fibre is 0.000008 m.'

Modern computers and calculators help these people to work with such numbers, but the numbers can still be complicated to write out, to use and to compare. The numbers may also be too long for a calculator or spreadsheet to cope with and therefore need to be written in a shorter form.

Very small and very large numbers can be expressed in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer. This is called **standard form** or **scientific notation**.

This notation is used to write numbers in a form that makes them:

- shorter
- easier to understand
- easier to compare
- able to fit onto a GDC or into a cell in a spreadsheet.

How to write a number in standard form

Large numbers

1. To write a very large number in standard form, first write down a value between 1 and 10 by inserting a decimal point after the first significant digit, e.g. for 13 000 write 1.3 (0.13 is less than 1, while 13.0, 130.00, etc. are all greater than 10).
2. Think about how many times you would need to **multiply** this new number by ten in order to get back to the original value, e.g. $1.3 \times 10 \times 10 \times 10 \times 10 = 13\,000$, so you need to multiply by 10 four times.
3. Then, rewrite the original number as the decimal value from step 1 multiplied by 10 raised to the power of the number of times you needed to multiply by 10, e.g. $13\,000 = 1.3 \times 10^4$.

For example, to write 5 120 000 in standard form:

5.12 is between 1 and 10.

$$5\,120\,000 = 5.12 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

5.12 needs to be multiplied by 10 six times to get back to the original number.

$$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$$

Multiplying by 10 six times can be written as 10^6.

So, 5 120 000 written in standard form is 5.12×10^6.

Another method is use the structure of the number in terms of place order:

10^6 (millions)	10^5 (hundreds of thousands)	10^4 (tens of thousands)	10^3 (thousands)	10^2 (hundreds)	10^1 (tens)	10^0 (units)
5	1	2	0	0	0	0

The first significant figure **5** has a place order in the millions and so indicates **5** million. It is in the 10^6 column, and this is the power of ten by which you need to multiply the value between 1 and 10.

Small numbers

1. To write a very small number in standard form, first write down a value between 1 and 10 by inserting a decimal point after the first significant digit (as you did for very large numbers), e.g. for 0.034 write 3.4.
2. Think about how many times you would need to **divide** the new number by 10 in order to get back to the original number, e.g. $3.4 \div 10 \div 10 = 0.034$, so you divide by 10 two times.
3. Then, rewrite the original number as the decimal value from step 1 multiplied by 10 raised to **negative** the number of times you need to divide by 10, e.g. $0.034 = 3.4 \times 10^{-2}$.

For example, to write 0.00684 in standard form:

$$\begin{aligned} 0.00684 &= 6.84 \div 10 \div 10 \div 10 \\ &= 6.84 \div 10^3 \\ &= 6.84 \times 10^{-3} \end{aligned}$$

Alternatively, use the place order structure of the number:

10^0 (units)	. (decimal point)	10^{-1} (tenths)	10^{-2} (hundreths)	$\mathbf{10^{-3}}$ (thousanths)	10^{-4} (tenths of thousandths)	10^{-5} (hundredths of thousandths)
0	.	0	0	**6**	8	4

The first significant figure **6** represents **6** thousandths. It is in the $\mathbf{10^{-3}}$ column, and this gives you the power of ten that you need.

Learning links

1B Negative indices and the standard form

In a number written in the form b^p, the superscript p is the **power** of b, also called the **index** or **exponent**. If p is a positive integer, it tells you the number of times to multiply by the number b. For example, $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

Negative powers

If p is a **negative** integer, it tells you that you have a reciprocal, or $\frac{1}{b^p}$. Recall that a reciprocal is a fraction with 1 in the numerator. For example, $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$ = 0.0625. Observe that

$$\begin{aligned} 1 \div 2 \div 2 \div 2 \div 2 &= \frac{1}{2} \div 2 \div 2 \div 2 \\ &= \frac{1}{2 \times 2} \div 2 \div 2 = \frac{1}{2 \times 2 \times 2} \div 2. \\ &= \frac{1}{2 \times 2 \times 2 \times 2} = \frac{1}{2^4} \end{aligned}$$

So 2^{-4} is the same as the fraction $\frac{1}{2^4}$.

This is an example of one of the laws of indices: $b^{-m} = \frac{1}{b^m}$.

How does this apply to the standard form?

The standard form is $a \times 10^k$ where $1 \le a < 10$ and k is an integer, which means that you have to **multiply** a by 10 raised to a power. Any small number that is less than 1 can always be written as $a \div 10^m$ or $a \times \frac{1}{10^m}$ where $1 \le a < 10$ and m is a **positive** integer. Using the law of indices above, this is the same as $a \times 10^{-m}$. In other words, we multiply by a **negative** power of 10 to indicate that we are really dividing by a power of 10.

Changing a number from standard form to ordinary form (decimal form)

To write a **large** number given in standard form $a \times 10^k$ as an ordinary number, **multiply** a by ten k times. For example:

$$7.904 \times 10^3 = 7.904 \times 10 \times 10 \times 10 = 7904$$

Here, $a = 7.904$ and $k = 3$. So multiply 7.904 by 10 three times (which is the same as multiplying by 1000).

To write a **small** number given in standard form $a \times 10^{-m}$ as an ordinary number, **divide** a by ten m times. For example:

$$3.816 \times 10^{-5} = 3.816 \div 10 \div 10 \div 10 \div 10 \div 10 = 0.00003816$$

Here, $a = 3.816$ and $m = 5$. So divide 3.816 by 10 five times (which is the same as dividing by 100 000).

It is possible to set your calculator to do all its calculations in standard form. Every time you use your GDC the screen will display the answer in the form that you have set. This can be useful if you often use very large or very small numbers, but not when you are doing calculations with more ordinary-sized numbers.

See '*1.2 Answers in standard form*' on page 650 of the GDC chapter to find out how to do operations in the form $\boldsymbol{a \times 10^k}$ with your GDC.

If a calculation is to be done in the standard form $a \times 10^k$, make sure that you know which key to use on your GDC to enter the exponent k. It will be marked **EE** or **EXP**.

Many people use numbers that are difficult to imagine and beyond their everyday experience. What is the largest number that you can really imagine? A hundred, 250, 1000, a million? What is the smallest number? One half, one tenth, one thousandth? Does using standard form make it easier for you, or more difficult, to grasp the magnitude of a number?

exam tip

Be careful not to write 'calculator language' in your working or answer rather than true mathematical notation. For instance, do not put 8.12E6 in your solution; instead, write 8.12×10^6. In examinations you will be expected to give answers in correct mathematical notation.

Worked example 1.10

Q. (a) If $a = 723\,000\,000$ and $b = 0.0591$, write a and b in standard form.

(b) Using the results from part (a) and your GDC, calculate:

(i) $a \times b$ (ii) $a \div b$ (iii) $b \div a$

Give your answers in standard form to three significant figures.

A. (a) $a = 723\,000\,000 = 7.23 \times 10^8$

7.23 is multiplied by 10 eight times.

$b = 0.0591 = 5.91 \times 10^{-2}$

5.91 has been divided by 10 twice.

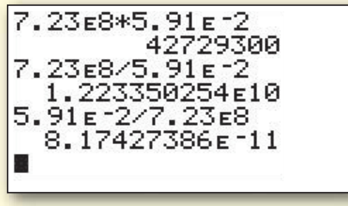
If numbers are very big or very small, the GDC gives the answers in standard form automatically.

Make sure you write the answers down using the correct notation! You would not get any marks if you write 8.17E–11.

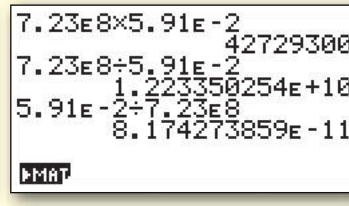
continued . . .

(b)

TEXAS



CASIO



(i) 4.27×10^7

(ii) 1.22×10^{10}

(iii) 8.17×10^{-11}

Exercise 1.7

1. Fill in the missing values of n, e.g. if $34\,500 = 3.45 \times 10^n$, then $n = 4$.

(a) $628 = n.28 \times 10^2$ (b) $53\,000 = 5.3 \times 10^n$

(c) $0.00282 = 2.82 \times 10^n$ (d) $3\,640\,000 = 3.n4 \times 10^6$

(e) $0.0208 = 2.n8 \times 10^{-2}$

hint

By 'decimal form' we mean an 'ordinary number'; so you need to write the number out in full as it would appear if not written in standard form. For example, 1.25×10^4 in 'decimal form' would be 12 500.

2. Write out the following numbers in decimal form.

(a) 1.25×10^4 (b) 3.08×10^3 (c) 2.88×10^8

(d) 4.21×10^{-2} (e) 9.72×10^{-3} (f) 8.38×10^{-6}

3. Write these numbers in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.

(a) 62 100 (b) 2100 (c) 98 400 000 (d) 52

4. Write these numbers in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.

(a) 0.727 (b) 0.0319 (c) 0.00000257 (d) 0.000408

5. Look at the following numbers:

398×10^1 0.17×10^3 2.4×10^{-3} 3.8×10^{-5}

370×10^2 0.02×10^2 1.2×10^2

(a) Which numbers are **not** written in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer?

(b) Rewrite the numbers from part (a) in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.

(c) Put all the numbers in ascending (increasing) order.

6. Use your GDC to calculate the following. Give each answer in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer. Round 'a' to three significant figures.

   For example, $(3.81 \times 10^{-2})^2 = 0.00145161 = 1.45 \times 10^{-3}$ (3 s.f.)

   (a) $(8.5 \times 10^3) \times (3.73 \times 10^6)$ (b) $(5.997 \times 10^2) \div (6.063 \times 10^3)$

   (c) $(7.71 \times 10^{-2}) \div (1.69 \times 10^7)$ (d) $(1.24 \times 10^{-3})^2$

   (e) $\sqrt{(6.59 \times 10^9)}$ (f) $(7.01 \times 10^{-3})^3$

7. Given that $m = 3.9 \times 10^3$ and $n = 6.8 \times 10^{-4}$, calculate the following, giving your answer in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer.

   (a) $m + n$ (b) $m - n$ (c) $m \times n$ (d) $m \div n$

8. If the speed of light is approximately 300 000 km s^{-1} and time $= \frac{\text{distance}}{\text{speed}}$, calculate the time that light takes to get from:

   (a) Earth to Jupiter, when they are 5.88×10^8 km apart

   (b) Mars to Venus, when they are 2.22×10^8 km apart

   (c) Saturn to the sun, when they are 1.43×10^9 km apart.

   Give your answers to the nearest minute. As all the figures have been rounded, these answers are only estimates.

9. The area of the Taman Negara park in Malaysia is 4.34×10^3 km^2. The area of Central Park in New York is 3.41×10^6 m^2. How many times can you fit Central Park into Taman Negara?

1.4 SI units

The international system of units (Système International d'Unités), or **SI units**, was adopted in 1960. The system is based on seven essential, or 'base', units for seven 'base' quantities that are independent of each other. The SI system forms a fundamental part of the language of science and commerce across the world.

Base quantity	Base unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Intensity of light	candela	cd

Time

The SI unit for time is seconds, but it is often convenient to work in minutes, hours, days, weeks, months or years.

Learning links

1C Working with decimals and time

It is important to remember that hours and minutes work in parts of 60, not parts of 10 or 100. Therefore, when we say '25.6 minutes', the '.6' does not represent 6 seconds. To convert 0.6 minutes to a number of seconds, think of it as 0.6 of a minute, that is, 0.6 of 60 seconds. So, multiply 60 by 0.6 to get $0.6 \times 60 = 36$ seconds.

Worked example 1.11

Q. Fiona works for herself and needs to earn \$20,000 per year.

(a) How much does she have to earn each month?

(b) How much does she have to earn each week?

(c) If she actually works 1000 hours per year, how many minutes is this? How many seconds?

A. (a) $\$20000 \div 12 = \$1666.67$ (to 2 d.p.)

There are 12 months in a year, so divide her yearly earnings by 12. For currencies it makes sense to round to 2 decimal places.

(b) $\$20000 \div 52 = \$384.62$ (to 2 d.p.)

There are 52 weeks in a year, so divide her yearly earnings by 52.

(c) $1000 \times 60 = 60\,000$ minutes

There are 60 minutes in an hour, so multiply the number of hours by 60 to get the number of minutes.

$60\,000 \times 60 = 3\,600\,000$ seconds

There are 60 seconds in a minute, so multiply the number of minutes by 60 to get the number of seconds.

Exercise 1.8

1. Convert these units of time:

   (a) 6 minutes 35 seconds to seconds

   (b) 562 seconds to minutes and seconds

   (c) 78 hours to days and hours

   (d) 6500 seconds to hours, minutes and seconds

   (e) 12 days 5 hours and 15 minutes to minutes

   (f) 6 hours, 7 minutes and 10 seconds to seconds

Temperature

The SI unit for temperature is the **kelvin** (K). This is the most important **scientific** measure of temperature, although in everyday life we use Celsius (°C) or Fahrenheit (°F). One kelvin has the same magnitude as one degree Celsius, but 0 K is at −273.15°C.

	Kelvin	Celsius	Fahrenheit
Freezing point of water	273.15	0°	32°
Boiling point of water	373.15	100°	212°

To convert a temperature in kelvins (K) to a temperature in Celsius (°C), use the formula:

$$T_C = T_K - 273.15$$

To convert a temperature in Celsius (°C) to a temperature in Fahrenheit (°F), use the formula:

$$T_F = \frac{9}{5}T_C + 32$$

To convert a temperature in Fahrenheit (°F) to a temperature in Celsius (°C), use the formula:

$$T_C = \frac{5}{9}(T_F - 32)$$

Worked example 1.12

Q. (a) The weather forecast in Houston, Texas, tells you that today's temperature will be 90°F. Give this temperature in degrees Celsius.

(b) The temperature in Stockholm on 1 January was −5°C. Give this temperature in degrees Fahrenheit.

Substitute $90 = T_F$ into the formula for converting Fahrenheit to Celsius.

Substitute $-5 = T_C$ into the formula for converting Celsius to Fahrenheit.

continued . . .

A. (a) $T_C = \frac{5}{9}(T_F - 32)$

$T_C = \frac{5}{9}(90 - 32) = 32.2\,°\text{C}$

(b) $T_F = \frac{9}{5}T_C + 32$

$T_F = \frac{9}{5} \times (-5) + 32 = 23\,°\text{F}$

Exercise 1.9

1. The table gives temperatures from cities around the world. The temperatures were recorded in January. Some are in measured in degrees Celsius and others in degrees Fahrenheit. Convert the units and complete the table.

City	Miami	Riga	Milan	Bahrain	Lima	Perth	Moscow
Celsius (°C)		−2	7		25		−12
Fahrenheit (°F)	82			65		90	

Other units

You also need to be able to use units that are closely related to the SI base units; these units are not official SI units, but are very commonly used.

Length	Mass	Volume
millimetre (mm)	milligram (mg)	millilitre (ml), 1 cm^3
centimetre (cm)		
metre (m)	gram (g)	
kilometre (km)	kilogram (kg)	litre (l), 1000 cm^3
	tonne (t), 1000 kg	

Is the fact that these prefixes are derived from Latin and Greek part of what makes mathematics a ‘universal language’? Or does the use of the Système International in mathematics seem more relevant to this point?

You will find it easier to use these units if you remember the meaning of the prefixes milli- and kilo-:

- ‘mille’ means one thousand in Latin; a millimetre is $\frac{1}{1000}$ of a metre
- ‘centum’ means one hundred in Latin; a centimetre is $\frac{1}{100}$ of a metre
- ‘khilloi’ means one thousand in Greek; a kilogram is 1000 grams, a kilometre is 1000 metres.

Derived units

There is no unit for volume among the seven SI base units. Volume is an example of a derived quantity with a **derived unit**; a unit defined in terms of the SI base units by means of a formula. The most common derived units are the following:

Derived quantity	Derived unit	Symbol
Area	square metre	m^2
Volume	cubic metre	m^3
Speed/velocity	metre per second	$m\,s^{-1}$
Acceleration	metre per second per second	$m\,s^{-2}$
Density	kilogram per cubic metre	$kg\,m^{-3}$

Speed, acceleration, force and density are all examples of quantities we come across in everyday life that use derived units. For example, evaluation of the force created by acceleration is important in crash testing and vehicle safety.

You can often work out what formula was used to generate the derived units. For example, the unit of 'area' is square metre ($m^2 = m \times m$), and 'm' is the SI unit for 'length', so area = length × length was the formula used.

When you use derived units, you are actually using simple formulae. For example:

$$\text{speed } (\text{m s}^{-1}) = \frac{\text{distance (m)}}{\text{time (s)}}$$

$$\text{density } (\text{kg m}^{-3}) = \frac{\text{mass (kg)}}{\text{volume (m}^3}$$

Worked example 1.13

Q. (a) Jared cycles 38 km in 3 hours and 20 minutes. What is his speed?

(b) If Jared continues at this speed, how long will it take him to cycle another 25 km? Give your answer to the nearest minute.

First convert the 20 minutes to hours to get the time in the same units: $\frac{20}{60} = \frac{1}{3}$. Then substitute values into the formula for speed.

A. (a) $\text{speed} = \frac{\text{distance}}{\text{time}}$

Jared's speed $= 38 \div 3\frac{1}{3} = 11.4\text{ km h}^{-1}$

Rearrange the formula $\text{speed} = \frac{\text{distance}}{\text{time}}$ to calculate the time from distance and speed. Then substitute in the distance given in the question and your answer from part (a).

(b) $\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{25}{11.4} = 2.193$

0.193 hours $= 0.193 \times 60 \approx 12$ minutes, so it will take 2 hours 12 minutes.

Exercise 1.10

1. Using the formula speed $= \frac{\text{distance}}{\text{time}}$, calculate:

(a) the speed of a car that travels 95 km in $2\frac{1}{2}$ hours

(b) the number of hours it takes to travel 37.4 km at 5.2 km h^{-1}

(c) the distance travelled if you go at a speed of 11 km h^{-1} for 2 hours 40 minutes.

2. (a) On the first day of their trek, Noah and his friends travel 21 km in eight hours. What is their average speed?

(b) On the second day they travel $18\frac{1}{2}$ km in seven hours. Are they travelling faster or slower on the second day?

(c) What is their average speed over the whole two-day trek?

Conversion between different units

If you are asked to convert from one unit to another, look at the units very carefully.

Remember that you cannot convert units of length into units of volume, or units of mass into units of temperature. You can only convert between units of the same *type* of measure. So you can change a length measure in one unit to a length measure in another unit, e.g. mm to km, and so on.

Most conversions will rely on you knowing how to get from one unit to another. The conversion of some common units are useful to know.

÷ 100	÷ 10	÷ 1000
1 m = 100 cm	1 cm = 10 mm	1 m = 1000 mm
× 100	× 10	× 1000

÷ 1000	÷ 60	÷ 60
1 kg = 1000 g	1 hour = 60 minutes	1 minute = 60 seconds
× 1000	× 60	× 60

Sometimes, the units given can guide you to the method you should use to convert them.

Worked example 1.14

Q. (a) The mass of a child is 21 kg. Give the child's mass in standard form in grams.

(b) A computer program runs for 4000 s. Give the time in hours, minutes and seconds.

(c) The area of a circle is 5.25 m^2. Give the answer in cm^2.

continued . . .

A. (a) 1 kg = 1000 g

> To get from kg to g, multiply by 1000.

$21\text{ kg} = 21 \times 1000\text{ g} = 21\,000\text{ g}$

> Convert into standard form.

$21\,000 = 2.1 \times 10^4$

so the child's mass is 2.1×10^4 g

> To convert seconds to minutes, divide by 60.

(b) $4000 \div 60 = 66.667$ minutes

> To convert 66 minutes to hours, divide by 60.

$66 \div 60 = 1.1$ hours

> To convert a fraction of an hour (0.1) to minutes, multiply by 60.

$0.1 \times 60 = 6$ minutes

> To convert a fraction of a minute (0.667) to seconds, multiply by 60.

$0.667 \times 60 = 40$ seconds

> See Learning links 1C on page 28 for a reminder about decimals and time.

So 4000 s = 1 hour, 6 minutes, 40 seconds

> To convert from m to cm, multiply by 100.

(c) 1 m = 100 cm

> Remember that to calculate an area, you need to multiply two lengths together. Each of these lengths must be converted to centimetres, so you need to multiply by 100 twice.

$1\text{ m}^2 = (100\text{ cm})^2 = 100 \times 100\text{ cm}^2$

$5.25\text{ m}^2 = 5.25 \times 100 \times 100 = 52\,500\text{ cm}^2$

In part (b) of Worked example 1.14, you could have used your GDC to convert 4000 s into hours, minutes and seconds:

TEXAS

```
4000/3600
     1.111111111
Ans▸DMS
         1°6'40"
```

CASIO

```
4000÷3600
               1.111111111
4000÷3600
                  1°06'40"
```

See '*1.3 Time in hours, minutes and seconds*' on page 651 of the GDC chapter for a reminder of how to do this, if you need to.

Worked example 1.15

Q. Esmé has a wooden box that measures $24\,cm \times 15\,cm \times 11\,cm$. Calculate the volume of the box in:

(a) mm^3 (b) cm^3 (c) litres

Convert to mm before doing the calculation: 1 cm = 10 mm, so multiply each measurement by 10.

A. (a) Method 1:

$240\,mm \times 150\,mm \times 110\,mm$

$= 3\,960\,000\,mm^3$

Work out the answer in cm^3 first, then convert to mm^3. To change cm^3 to mm^3, you need to multiply by 10 three times (as you multiply three lengths when calculating volume).

Method 2:

$24\,cm \times 15\,cm \times 11\,cm$

$= 3960\,cm^3 = 3960 \times (10\,mm)^3$

$= 3960 \times 10 \times 10 \times 10 = 3\,960\,000\,mm^3$

The lengths are already in cm, so there is no need to convert.

(b) $V = 24\,cm \times 15\,cm \times 11\,cm = 3960\,cm^3$

$1000\,cm^3$ = 1 litre, so divide the answer from (b) by 1000.

(c) $V = 3960 \div 1000 = 3.96$ litres

Exercise 1.11

In questions 1–5, convert the units as instructed.

For example, if the question says '5.75 litres to ml', give the answer as '5750 ml'.

1. (a) 3500 mm to m (b) 276 cm to mm
   (c) 4800 m to km (d) 352 m to cm
2. (a) 5.8 kg to g (b) 30 g to kg
   (c) 1260 mg to g (d) 1 kg to mg
3. (a) $4.5\,m^2$ to cm^2 (b) $685\,cm^2$ to m^2
   (c) $1.4\,km^2$ to m^2 (d) $120\,mm^2$ to cm^2
4. (a) $12\,m^3$ to cm^3 (b) $24\,000\,cm^3$ to m^3
   (c) $1.3\,cm^3$ to mm^3 (d) $0.5\,m^3$ to cm^3
5. (a) 7900 ml to litres (b) 3.95 litres to ml
   (c) 83.3 litres to cm^3 (d) 687 ml to cm^3

6. (a) A rectangle has a length of 2.5 m and a width of 2.8 m. Give the area of the rectangle in (i) cm^2; (ii) m^2.

(b) A cylinder has a height of 1.29 m and a radius of 45 cm. Using the formula $V = \pi r^2 h$, give the volume of the cylinder in (i) cm^3; (ii) m^3.

(c) A triangle has a base of length 12.5 cm and a height of 53 mm. What is the area of the triangle in cm^2?

7. You have 7 m of ribbon that you want to cut into lengths of 18 cm.

(a) How many pieces of ribbon will you have?

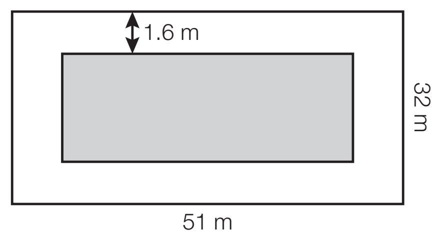
(b) How much ribbon will you have left over?

8. You want to buy tiles to cover an area 3 m long and 1.5 m wide. You choose tiles that are 6 cm square. How many tiles will you need to buy?

9. A large bucket has a volume of $8\frac{1}{2}$ litres. You need to measure out cupfuls of 160 cm^3.

(a) How many cupfuls will you measure?

(b) How much liquid will you have left over?



10. (a) Timmi is making a path round the edge of her garden. Calculate the area of the path (the non-shaded region).

(b) The concrete path is 8 cm deep. Calculate the volume of concrete that Timmi needs in cubic metres.

Summary

You should know:

- the definitions of the natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$ and real numbers $\mathbb{R}$
- how to round numbers to a given number of decimal places or significant figures, and that this process is called approximation
- how to calculate percentage error using the formula
$$\varepsilon = \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$$
where v_E is the exact value and v_A is the approximate value
- how to sensibly estimate the answer to a calculation, and how to use your estimate to check solutions obtained from a GDC or computer
- how to express numbers in the standard form $a \times 10^k$ where $1 \le a < 10$ and k is an integer, and how to perform calculations with numbers in this form
- what the SI (Système International) is and give examples of these units as well as other basic units of measurement, such as kilogram (kg), metre (m), second (s), litre (l), metres per second ($m\,s^{-1}$) and degrees Celsius (°C).

Mixed examination practice

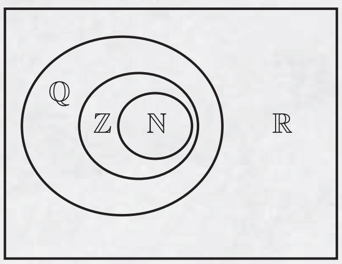
Exam-style questions

1. Copy and complete the table. One box has been completed for you.

	11	$\frac{1}{11}$	$\sqrt{11}$	−11
$\mathbb{N}$				
$\mathbb{Z}$				
$\mathbb{Q}$				
$\mathbb{R}$			✓	

2. Put the following numbers into the correct set in the given Venn diagram:

π, 12, sin (30°), 0.02, 0, –2



GDC

To calculate sin (30°) on your GDC, you do not need to use brackets; just press [sin] [3] [0].

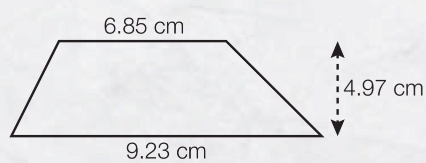
3. Calculate each the following and decide which type of number the answer is. For example, $(14+5)\div 7 = 2\frac{5}{7}$ is a rational number.

(a) $(6+8)\div(9-2)$ (b) $6+(8\div 9)-2$ (c) $2^3\times 2^6\div 2^8$

(d) $2\times\pi\times 6.8$ (e) $(\sqrt{2})^2+(\sqrt{3})^2+(\sqrt{5})^2$ (f) 40% of –90

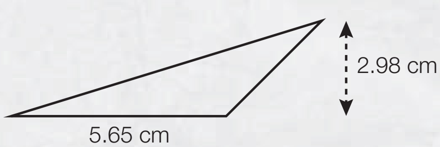
(g) $3(8-\frac{1}{3})-5(12-\frac{2}{3})$ (h) If $6(x+3)=x-2$ then $x=?$

4. In this question, give your answers to one decimal place.

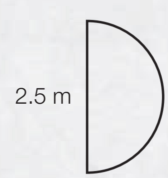


(a) Using the formula $A=\frac{1}{2}(a+b)h$, where a and b are the lengths of the parallel sides and h is the height, calculate the area of the trapezium.

(b) Using the formula $A=\frac{1}{2}(b\times h)$, where b is the base and h is the height, calculate the area of the triangle.



(c) Using the formula $C=2\pi r$, where r is the radius, calculate the perimeter of the semi-circle.



5. Suppose that $A = \pi(R^2 - r^2)$, where $R = 19.29$ and $r = 11.01$.

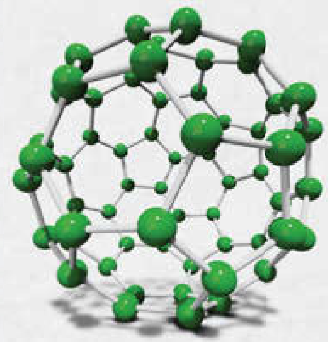
   (a) Estimate the values of R, r and π to one significant figure.

   (b) Use your answers in (a) to estimate a value for A.

   (c) Calculate an accurate value for A, using the exact values for R and r. Give your answer to 3 s.f.

6. Calculate $\dfrac{1+\sqrt{8.01}}{1.2^2}$, giving your answer to (i) one decimal place; (ii) three significant figures.

7. Given that the diameter of a molecule of C_{60} (buckminsterfullerene) is 1.1×10^{-9} m, how many molecules would fit along a line 1 cm long?


Buckminsterfullerene.

8. Use the formula density $= \dfrac{\text{mass}}{\text{volume}}$ to calculate the following.
   Include units in your answers.

   (a) 1 m^3 of plastic weighs 958 kg. What is its density?

   (b) Change 1 m^3 into cm^3.

   (c) What is the mass of 10 cm^3 of this plastic?

9. Osmium is the densest known solid. The density of osmium is 22 610 kg m^{-3}.

   (a) What is the mass of 0.5 m^3?

   (b) What is the volume of a piece with a mass of 100 g?

Past paper questions

1. A problem has an **exact** answer of $x = 0.1265$.

   (a) Write down the **exact** value of x in the form $a \times 10^k$ where k is an integer and $1 \le a < 10$.

   (b) State the value of x given correct to **two** significant figures.

   (c) Calculate the percentage error if x is given correct to **two** significant figures. *[Total 6 marks]*

[May 2006, Paper 1, Question 3] (*© IB Organization 2006*)

2. (a) Calculate $\dfrac{77.2 \times 3^3}{3.60 \times 2^2}$. *[1 mark]*

   (b) Express your answer to part (a) in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$. *[2 marks]*

   (c) Juan estimates the length of a carpet to be 12 metres and the width to be 8 metres.
   He then estimates the area of the carpet.

      (i) Write down his estimated area of the carpet. *[1 mark]*

      When the carpet is accurately measured it is found to have an area of 90 square metres.

      (ii) Calculate the percentage error made by Juan. *[2 marks]*

[Total 6 marks]

[Nov 2007, Paper 1, Question 1] (*© IB Organization 2007*)

Chapter 2 Solving equations

In this chapter you will learn:

- how to use a GDC to solve linear equations with one variable
- how to use a GDC to solve pairs of linear equations with two variables
- how to use a GDC to solve quadratic equations.

Algorithms are the building blocks of the modern world. Examples of algorithms include: the fundamental instructions for a computer; the procedure that a pharmaceutical firm follows in researching a new drug; and the strategy that a company uses to organise deliveries of letters and parcels all over the world.

The words 'algorithm' and 'algebra' both come from the work of the same mediaeval Arab mathematician, Al-Khwarizmi, who was a scholar at the House of Wisdom in Baghdad. Al-Khwarizmi wrote a textbook called *Hisab al-jabr w'al-muqabala*, which contained the first explanations of some of the techniques that are still used in modern algebra.

'Al-jabr' became the word **algebra**, a method of generalising problems in arithmetic.

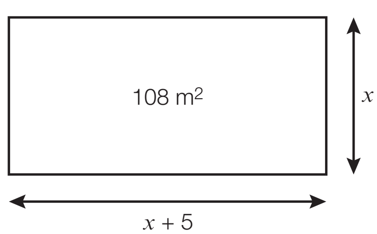
Al-Khwarizmi's own name became the word **algorithm**, a term describing a powerful scientific way of solving problems.

In this chapter you will meet some of the oldest and most frequently used algorithms: those used to solve linear and quadratic equations.

Teachers are often asked, 'Why do we have to learn how to solve quadratic equations?' This is a question that is already thousands of years old!

About 3500 years ago, Egyptians and Babylonians (in present-day Iraq) wanted to know how to calculate the sides of a square or rectangle with any given area; these problems involved quadratic equations that had to be solved to find the lengths that they needed.

In modern algebra, if you want to find the dimensions of a rectangle which has one side 5 m longer than the other and whose area is 108 m^2, you could create an equation from the formula for the area of a rectangle:



$$A = \text{length} \times \text{breadth}$$

$$x(x + 5) = 108$$

$$x^2 + 5x = 108$$

The Egyptians did not use algebraic methods like those above. They solved the problems using a set procedure — an algorithm. The solutions were collected in tables, so that the sages and engineers could look up solutions for different shapes and different areas.

Today, the problems are still very similar; but instead of using a prepared table to solve your equations, you can use algebra, a calculator or a computer.

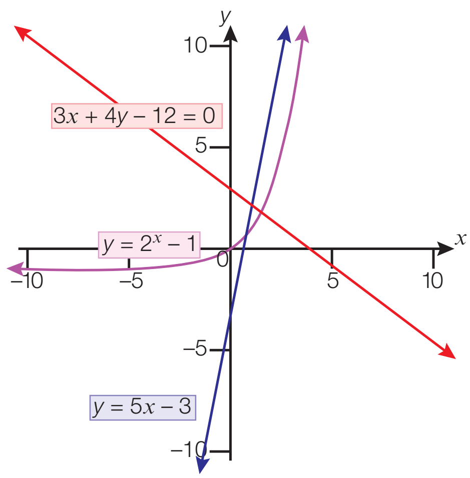
2.1 Linear equations

You can recognise a **linear equation** from its **general form**:

$$y = mx + c \qquad \text{or} \qquad ax + by + c = 0$$

The highest power of x (also called the 'order') in a linear equation is 1. When plotted on a graph, a linear equation takes the form of a straight line: as you can see on the graph below, the linear equations $y = 5x - 3$ and $3x + 4y - 12 = 0$ each give a straight line.

Plotted on the same graph is the equation $y = 2^x - 1$, which does **not** give a straight line. This is not a linear equation because it cannot be written in either of the general forms above; neither general form contains x as an exponent (a power). Make sure you do not confuse $2x$, which is linear, with 2^x, which is not.



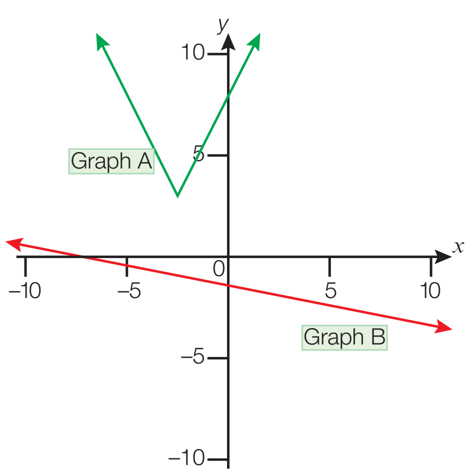
Exercise 2.1

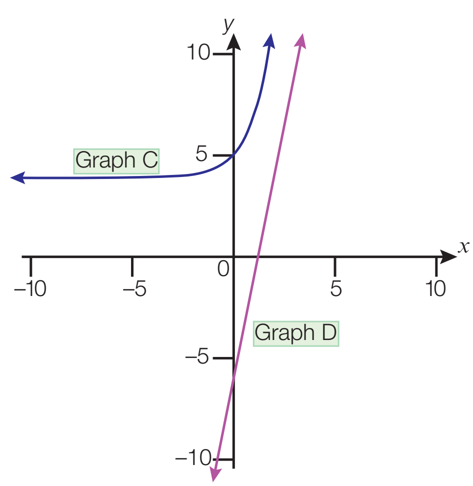
1. Identify which of these equations are linear equations and which are not.

(a) $y = 3x$ (b) $y = \frac{2}{x} + 4$ (c) $y = 7x^2 - 3$

(d) $x = 9 - 5y$ (e) $x - y - \frac{3}{4} = 0$ (f) $y = -8$

2. Which of these are graphs of linear functions?





Rearranging linear equations

You need to be able to rearrange linear equations into different forms. Rearranging an equation can be used to find its solution, and is a skill required so that you can enter equations into your GDC in the correct format.

Worked example 2.1

Q. (a) Rearrange $y = 5x - 3$ into the form:

(i) $y - mx = c$ (ii) $y - mx + c = 0$

(b) Rearrange $3x + 4y - 12 = 0$ into the form:

(i) $mx + by = c$

(ii) $by = mx + c$

(iii) $y = \dfrac{mx + c}{d}$

A. (a) (i) $y - 5x = -3$

Subtract $5x$ from both sides of the equation.

(ii) $y - 5x + 3 = 0$

Add 3 to both sides of the equation from (i).

(b) (i) $3x + 4y = 12$

Add 12 to both sides of the equation.

(ii) $4y = -3x + 12$

Subtract $3x$ from both sides of the equation from (i).

(iii) $y = \dfrac{-3x + 12}{4}$

Divide both sides of the equation from (ii) by 4.

Learning links

2A Things to remember when rearranging equations

To rearrange equations, you need to remember some important points about algebra:

1. Letters are used to represent unknown values; these are called the **variables**.
2. Each equation is made up of different **terms** that are separated by either a '+' or '–' operator or the '=' sign. The '×' and '÷' operators do **not** separate terms; they form part of the term.

→

continued . . .

3. To make a variable the 'subject' of the equation, you have to get that variable on its own on one side of the '=' sign.
4. You might need to apply the **inverse** operation ('undo it') to make the variable the subject of the equation.
5. You need to apply the same operations to both sides of the equation to keep it balanced.
6. Remember the **BIDMAS** order of operations: deal with brackets first (by expanding them), then indices (powers or exponents can be reversed by taking the equivalent root), then multiplication and division (if both are present, do them from left to right), and finally addition and subtraction (if both are present, do them from left to right).

Exercise 2.2

1. The following are all linear equations.
Rearrange them into the general form $y = mx + c$.

(a) $x - y + 4 = 0$ (b) $3x + y = 7$ (c) $x - 9y = 15$

(d) $5(3 - x) - 2y = 0$ (e) $11(x - y) = 10$ (f) $2(x - 3y - 4) = 5$

(g) $\frac{1}{2}x + \frac{2}{3}y = 9$ (h) $\frac{5x}{y-1} = \frac{4}{3}$ (i) $\frac{y-7}{x} = \frac{2}{9}$

2. The following are all linear equations.
Rearrange them into the form $ax + by + c = 0$.

(a) $y = 5x + 4$ (b) $y = \frac{1}{2}(x - 5)$ (c) $3(2 - x) = 2y$

(d) $3(x - 2) = 4(y + 1)$ (e) $\frac{2y}{3} = 3x$ (f) $\frac{1-5x}{2} = y$

Solving linear equations

Traditionally, people used graphs or algebra to solve linear equations. Today, you can solve linear equations using your GDC.

To gain confidence in using your GDC accurately, it is a good idea to try solving some simple equations using algebra first, and then check the answers using your GDC. See Learning links 2B on page 42 if you need a reminder of how to solve linear equations algebraically or graphically.

hint

When you are asked to **solve** an equation, you are being asked to find the numerical value that makes the equation true after you replace all instances of the letter with the value; such a value is called a **solution** to the equation.

Learning links

2B Solving linear equations using algebra

To solve an equation using algebra, remember the following points:

- When solving an equation you need to make the variable the subject.
- Make sure you keep the equation balanced by doing the same to both sides of the '=' sign.

$$\frac{4(x+3)}{2} = 24$$

In this example, 'x' is the variable, so we need to make this the subject, i.e. get it on its own.

$$4(x + 3) = (4 \times x) + (4 \times 3)$$

$$= 4x + 12$$

By the order of operations, we first need to expand (multiply out) the bracket on the lefthand side.

$$\frac{4x+12}{2} = 24$$

Multiply each term on both sides of the '=' sign by 2 (to get rid of the fraction).

$$4x + 12 = 48$$

Subtract 12 from both sides.

$$4x = 36$$

$$x = 9$$

Then divide both sides by 4.

So the solution is $x = 9$.

- Check that the solution is correct by substituting the value for x in the original equation:

$$\frac{4(9+3)}{2} = \frac{4 \times 12}{2} = \frac{48}{2} = 24$$

Substitute in $x = 9$.

Therefore the solution is correct.

2C Solving linear equations using a graph

You can solve a linear equation by plotting its graph, in the general form $y = mx + c$, on the a set of **axes** (**x-axis** and **y-axis**).

Suppose we wanted to find the solution of the equation $5x - 3 = 22$.

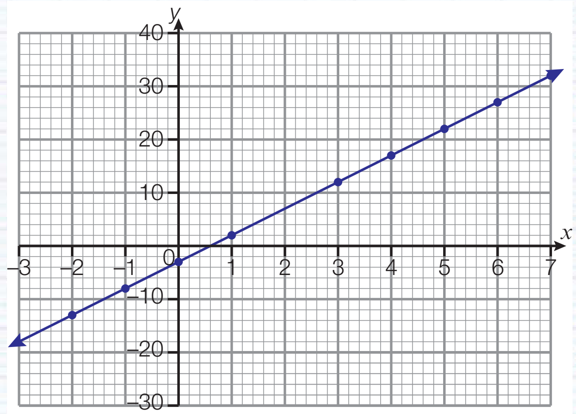
- Rewrite the equation in the general form $y = mx + c$

  and plot the graph of this equation:

$$y = 5x - 3$$

→

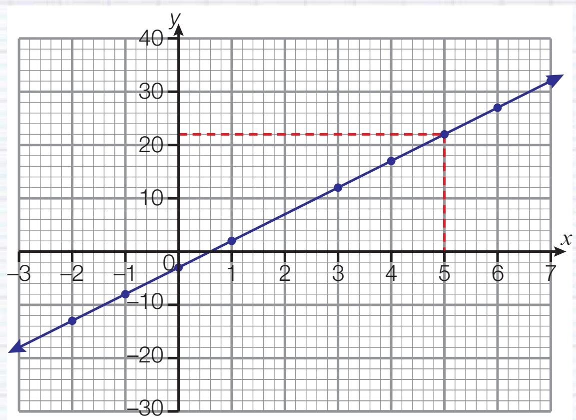
continued . . .



- Use the graph of $y = mx + c$ to find the value of x at which the original equation is true.

From the graph,
$y = 22$ when $x = 5$.
$5x - 3 = y = 22$

If $5x - 3 = 22$, and $y = 5x - 3$, then $y = 22$. Find 22 on the y-axis. At the point where $y = 22$ meets the graph of $y = 5x - 3$, read off the x-coordinate from the x-axis (you might find it helpful to draw the line $y = 22$). The x-coordinate is the solution.



$x = 5$

- Check your answer by substituting the value of x back into the original equation.

$5x - 3 = 22$

If $x = 5$, then

$5x - 3 = 5 \times 5 - 3$

$= 25 - 3 = 22$

The solution is correct.

Worked example 2.2

Q. Solve the following equations using algebra, and check your solutions using your GDC.

(a) $5x - 8 = -14$ (b) $\frac{1}{2}y + 12 = 2y - 3$

(c) $3(x + 1) = 5(x - 2)$ (d) $\frac{3 - 2x}{4} = 5$

A.

Add 8 to both sides then divide both sides by 5.

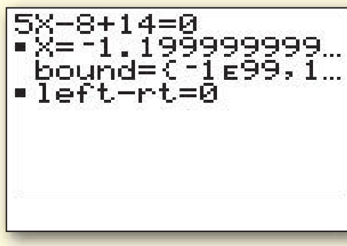
(a) $5x - 8 = -14$

$5x = -6$

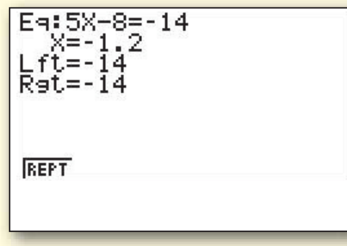
$x = -1.2$

Check the answer using your GDC. (See '*2.1(b) Solving linear equations using an equation solver*' on page 653 of the GDC chapter if you need a reminder).

TEXAS



CASIO



Add 3 to both sides, subtract $\frac{1}{2}y$ from both sides, then multiply by $\frac{2}{3}$.

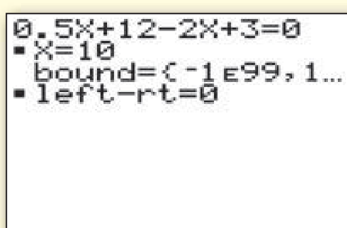
(b) $\frac{1}{2}y + 12 = 2y - 3$

$15 = \frac{3}{2}y$

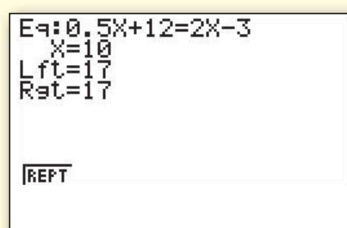
$10 = y$

Check the answer using your GDC.

TEXAS



CASIO



Expand the brackets on each side. Combine the terms containing the variable on one side of the equation, and move the constants to the other side.

(c) $3(x + 1) = 5(x - 2)$

$3x + 3 = 5x - 10$

$13 = 2x$

$6.5 = x$

continued . . .

Check the answer using your GDC.

TEXAS

```
3(X+1)-5(X-2)=0
▪X=6.5
 bound={-1E99,1…
▪left-rt=0
```

CASIO

```
Eq:3(X+1)=5(X-2)
  X=6.5
Lft=22.5
Rgt=22.5

REPT
```

Multiply both sides by 4 to get rid of the fraction first. Add $2x$ to both sides, then subtract 20 from both sides. Divide both sides by 2.

(d) $\frac{3-2x}{4}=5$

$3-2x=20$

$-17=2x$

$-8.5=x$

Check the answer using your GDC.

TEXAS

```
(3-2X)/4-5=0
▪X=-8.499999999…
 bound={-1E99,1…
▪left-rt=0
```

CASIO

```
Eq:(3-2X)÷4=5
  X=-8.5
Lft=5
Rgt=5

REPT
```

Exercise 2.3

Solve the following linear equations and check your answer using a GDC.

1. (a) $5m + 8 = 13$ (b) $0.2z - 12 = 2$

(c) $6 - 2y = 8$ (d) $3 - 0.5x = -5$

2. (a) $6m - 2 = 3m + 8$ (b) $5 - 2f = 3 + 8f$ (c) $7x - 4 = 10 + 2x$

3. (a) $5(x + 1) = 3(2 - x)$

(b) $0.4(z - 2) = 1.2(5 - z)$

(c) $-2(y + 5) = 3(3 - y)$

4. (a) $\frac{x+2}{5} = 3$ (b) $\frac{y-3}{2} = \frac{y}{5}$

(c) $\frac{m+1}{3} = \frac{4-m}{2}$ (d) $\frac{2x+5}{3} = \frac{3-4x}{2}$

2.2 Pairs of linear equations

In some situations you may have two **unknowns** that need to be found. If there are two conditions involving the unknowns, you can write an equation for each. For example:

Dean buys 3 chocolate bars and 4 cokes for \$12.

Poppy buys 5 chocolate bars and 1 coke for \$3.

Let x dollars be the price of each chocolate bar and y dollars the price of each coke drink. Then we know that:

$3x + 4y = 12$ and

$5x + y = 3$

This forms a pair of linear equations.

A **pair of linear equations** means 'two linear equations containing two values that you do not know'.

Pairs of linear equations are also called **simultaneous equations** because both equations are true at the same time (i.e. simultaneously).

Solving pairs of linear equations

You can solve pairs of linear equations, but instead of having to find the value of one unknown, you need to find the values of two unknowns. Again, traditional methods include drawing a graph or using algebra but you can also use your GDC. For a reminder of the more traditional methods, see Learning links 2D and 2E on page 47.

For most GDCs you will need both equations to be in their general form in order to use them (for example $y = mx + c$). Make sure you are confident in rearranging equations because your GDC cannot do it for you. (See Learning links 2A on page 10 if you need a reminder of how to rearrange equations.)

The link between algebra and geometry was developed by the French mathematician and philosopher René Descartes (1596–1650). His work enabled geometrical problems to be solved using algebra, and led to many other mathematical advances such as calculus.

exam tip

This course encourages you to solve pairs of linear equations using your GDC. In the examination, however, you can use whichever method you are most comfortable with.

Worked example 2.3

Q. (a) Use your GDC to plot the graph of each equation and hence solve the following pair of linear equations:

$y + x = 2$ and $y + 3x = 0$

A. (a) $y + x = 2$ $y + 3x = 0$

$y = 2 - x$ $y = -3x$

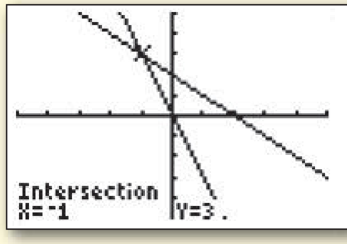
$y_1 = -x + 2$ $y_2 = -3x$

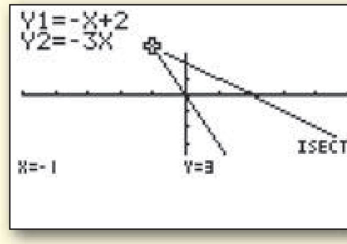
Rearrange the equations so that they are both in the general form $y = mx + c$.

continued . . .

Use your GDC to solve. See *'2.2 (a) Solving pairs of linear equations using a graph'* on page 653 of the GDC chapter if you need a reminder.

TEXAS | **CASIO**





Write down the answer in the appropriate format.

$x = -1$ and $y = 3$.

Q. (b) Use your GDC to solve the following pairs of linear equations using an algebraic method:

$5x + 2y = 13$ and $y = x - 4$

Rewrite each equation in the form $ax + by = c$.

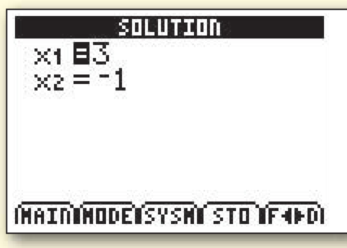
A. (b) $5x + 2y = 13$ (The equation is already in the required format.)

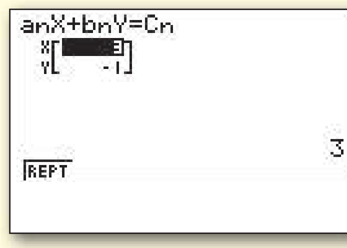
$y = x - 4$

$-x + y = -4$

Use your GDC to solve. See *'2.2 (b) Solving pairs of linear equations using an equation solver'* on page 655 of the GDC chapter if you need a reminder.

TEXAS | **CASIO**





Write down the answer in the appropriate format.

$x = 3$ and $y = -1$

Learning links

2D Solving pairs of linear equations using algebra

Using **elimination**, solve the equations $x + 2y = 12$ and $x - y = -3$.

1. Number each equation:

$x + 2y = 12$ (1)

$x - y = -3$ (2)

continued . . .

2. Subtract one equation from the other (or add them) to remove one of the unknowns.

In this example, (1) – (2) will eliminate x and leave us with just y.

$$x - x + 2y - (-y) = 12 - (-3)$$
$$3y = 15$$
$$y = 5$$

3. Substitute this value of y into one of the original equations to find the value of x.

Use equation (2).

$$y = 5 \Rightarrow x - 5 = -3$$
$$x = 2$$

4. So the solution is $x = 2$ and $y = 5$.

Using **substitution**, solve the equations $5x + 2y = 13$ and $y = x - 4$.

1. Number the equations as before:

$$5x + 2y = 13 \quad (1)$$
$$y = x - 4 \quad (2)$$

2. Substitute one equation into the other to eliminate one of the unknowns.

In this example, it is more convenient to substitute (2) into (1). So, in equation (1) 'y' is replaced by '$x - 4$'.

$$5x + 2(x - 4) = 13$$
$$5x + 2x - 8 = 13$$
$$7x = 13 + 8$$
$$7x = 21$$
$$x = 3$$

3. Substitute the value of 'x' into one of the original equations.

Here we have used equation (2).

$$x = 3 \Rightarrow y = x - 4 = 3 - 4 = -1$$

So the solution is $x = 3$ and $y = -1$.

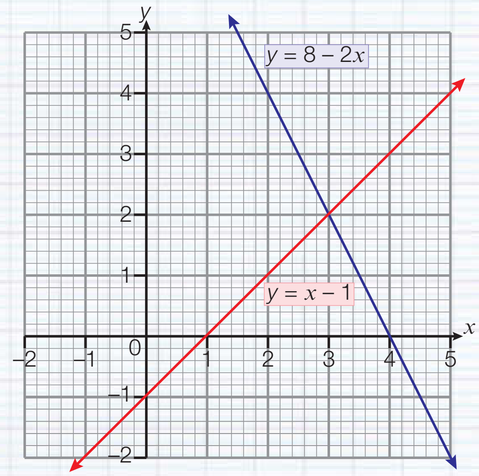
2E Solving pairs of linear equations using graphs

You can solve pairs of linear equations by plotting their graphs, in the general form $y = mx + c$, on the same set of **axes** (**x-axis** and **y-axis**).

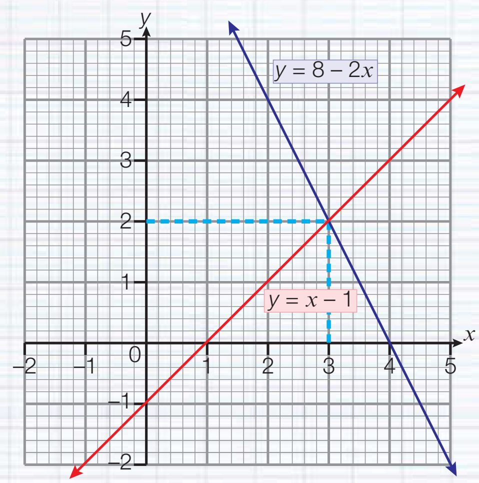
Suppose we wanted to solve the following pair of linear equations: $2x + y = 8$ and $x - y = 1$.

continued . . .

1. Plot them on the same set of axes.



2. Find the **intersection** of the lines (the point where the two lines cross); this is where the x and y values are the same for both equations. The x and y coordinates are the solutions to the equations, so $x = 3$ and $y = 2$.



3. Check the answer by substituting these values into the original equations:

$$2x + y = 8 \qquad \text{and} \qquad x - y = 1$$
$$2 \times 3 + 2 = 8 \qquad\qquad 3 - 2 = 1$$

Exercise 2.4

Solve the following pairs of linear equations with your GDC.

1. (a) $y = x + 5$ and $y = 2x - 3$

(b) $y = 3x - 5$ and $y = 10 - 2x$

(c) $y = 5 - 2x$ and $4x + 3y = 14$

(d) $2x + 3y = 42$ and $3x = 4y - 5$

(e) $5x + 3y + 11 = 0$ and $2x - 4y = 10$

(f) $2x + 7y - 9 = 0$ and $3x - 5y = 6$

(g) $y + 2 = x$ and $y = 1.8x - 4.88$

(h) $7x - 6y + 12 = 0$ and $4x + 3y - 9 = 0$

Do you think that it is easier to work out the solution to a problem if you can see it as a picture? Think about solving pairs of linear equations: which method is the clearest for you? Using algebra, graphs or technology (GDC)? After you have solved the problem, which method has given you the best understanding of the answer?

2. (a) $0.78s + 0.54t = 1.96$ and $3s - 2t = 23$

(b) $0.59s + \frac{1}{3}t = \frac{9}{7}$ and $1.7s = 3.1t + 8.4$

(c) $-8s + 4t + \frac{7}{6} = 0$ and $s - \frac{2}{3}t = \frac{8}{11}$

(d) $0.125s + 0.61t - 45 = 0$ and $\frac{1}{4}s - 1.15t = 11$

Solving practical problems involving pairs of linear equations

Pairs of linear equations can be used to solve practical problems that you might come across in everyday life.

Worked example 2.4

Q. One day, Kazuo orders three drinks and two pizzas from a pizza restaurant in town. The total cost comes to \$7.95. The next day, Kazuo orders five drinks and three pizzas. This time he pays \$12.42. He expects his friends to pay him back for these purchases. But what is the cost of one drink? And what is the cost of one pizza?

Read the question, and decide on letters for the unknown quantities that you are trying to find.

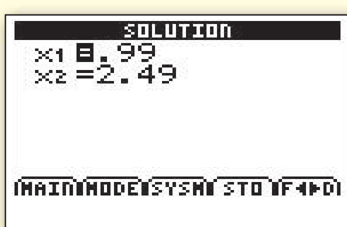
A. Let x be the cost of a drink and y the cost of a pizza (in dollars).

Translate the statements in the problem into algebraic equations: you know that on the first day Kazuo bought three drinks ($3x$) and two pizzas ($2y$) for \$7.95, so you can write this as an equation: $3x + 2y = 7.95$.

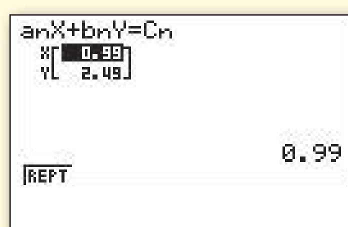
Equation for the first day: $3x + 2y = 7.95$
Equation for the second day: $5x + 3y = 12.42$

Solve the pair of equations using your preferred method; here we have used the equation solver on the GDC.

TEXAS



CASIO



Write down the answer.

continued . . .

$x = 0.99$ and $y = 2.49$

Kazuo can tell his friends that a drink cost \$0.99 and a pizza cost \$2.49.

Exercise 2.5

1. Two litres of milk and three loaves of bread cost 101 roubles. Three litres of milk and one loaf of bread cost 85 roubles.

   Using the information above, we can form the equation:

   $2m + 3b = 101$

   where m is the price of a litre of milk and b is the price of a loaf of bread in roubles.

   (a) Form another equation using the information from above.

   (b) Hence solve the pair of equations to find the values of m and b.

2. Zainab and Zahra bought some chocolates from the duty free shop in Dubai.

   Zainab bought 5 bars of Miniature Snickers and 3 bars of Twix Classics for a total of 126 AED.

   Zahra bought 6 bars of Miniature Snickers and 2 bars of Twix Classics for a total of 100 AED.

   Zainab's purchases can be represented by the equation:

   $5s + 3t = 126$

   where s is the price (in AED) of a Snickers bar and t is the price of a Twix bar.

   (a) Write an equation to represent Zahra's purchases.

   (b) Solve the pair of linear equations to find the price of each bar of chocolate.

3. The sum of two numbers is 97 and the difference between them is 23.

   By representing the larger number as x and the smaller one as y, write two separate equations for the sum and difference of the two numbers. Solve the equations to find both numbers.

4. Anton and Simone went to a music shop. They decided to buy the same CDs and DVDs. Anton bought five CDs and three DVDs for £104.92. Simone bought two CDs and seven DVDs for £128.91.

An equation for Anton's purchases can be written as:

$$5c + 3d = 104.92$$

where c and d are the unit costs in pounds of the CDs and DVDs, respectively.

(a) Write another equation to represent Simone's purchases.

(b) Hence solve the pair of equations to find the unit costs of the CDs and DVDs.

5. Three packs of batteries and a calculator cost £23.47. Five packs of batteries and seven calculators cost £116.45. Solve the appropriate simultaneous equations to find the costs of one pack of batteries and one calculator.

6. New Age Computers sells two models of laptop: easy-click and smooth-tab. Three easy-click and four smooth-tab laptops cost \$2987. Two easy-clicks and five smooth-tabs cost \$3123.

(a) Form a pair of linear equations to represent this information.

(b) Solve the equations to find the unit cost of each type of laptop.

7. Two adults and five children pay a total of \$120 for a coach journey. Three adults and seven children pay a total of \$172.50 for the same coach journey.

Find the total cost of tickets for eight adults and 11 children travelling on the same coach journey.

8. A mathematics test consists of shorter and longer questions. Each shorter question is worth 6 marks and each longer question is worth 11 marks.

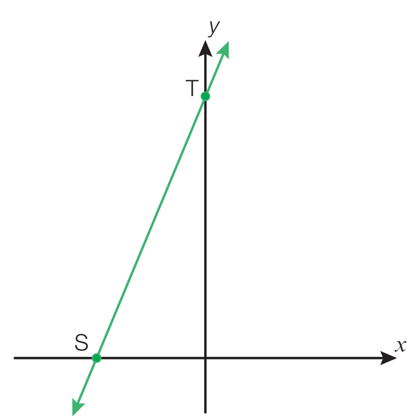
Mrs Pavlov sets a test with 15 questions and a total of 120 marks. By forming the appropriate equations and solving them, find the number of questions of each type on this test.

9. In the diagram below, the straight line ST, with equation $3x - y + 7 = 0$, intersects the coordinate axes at the points S and T.

(a) Find the coordinates of the points S and T.

A different straight line, with equation $2x + y + 3 = 0$, intersects the line ST at the point R.

(b) Find the coordinates of the point R.



2.3 Quadratic equations

Quadratic equations can be recognised by the x^2 term, which they always contain.

The general form of a **quadratic equation** is $ax^2 + bx + c = 0$, where $a \neq 0$ (but b and c can be zero).

hint

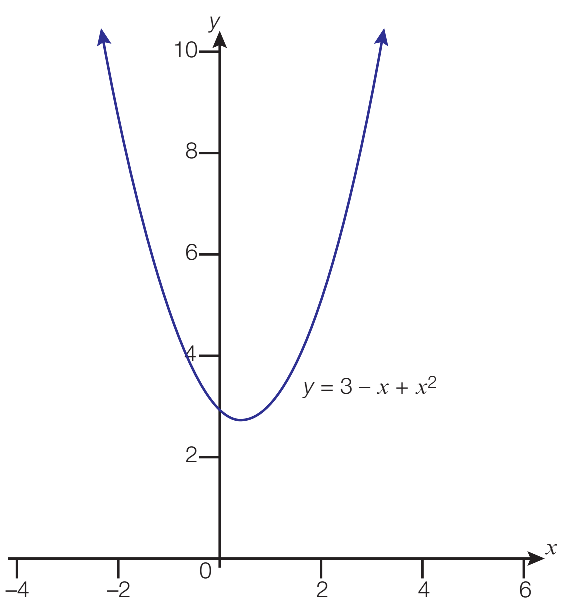
Remember that ≠ means 'is not equal to' or 'does not equal'.

For example, the following equations are quadratic:

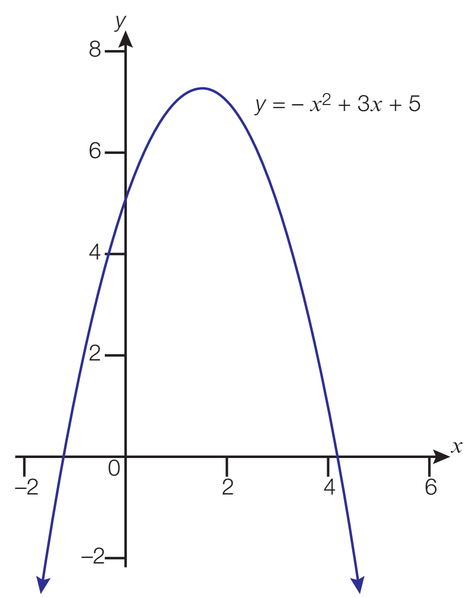
$2x^2 + 3x - 4 = 0$	The equation has $a = 2$, $b = 3$ and $c = -4$.
$x^2 - 4x = 0$	The equation has $a = 1$, $b = -4$ and $c = 0$.
$4x^2 - 49 = 0$	The equation has $a = 4$, $b = 0$ and $c = -49$.

When plotted on a graph, a quadratic equation $y = ax^2 + bx + c$ forms a curve that has one turning point and a central line of symmetry. This type of curve is called a **parabola**.

To plot the graph of a quadratic equation, it needs to be rearranged into the general form $y = ax^2 + bx + c$. If a is positive, the curve will have a minimum point; if a is negative, the curve will have a maximum point.



Minimum point because a is positive



Maximum point because a is negative

A parabola is a very important shape.

- If you throw a ball, the ball takes the path of a parabola; you can use this fact to predict the best angle at which to serve a tennis ball.
- Satellite dishes are parabolic, because this shape concentrates all the radio waves into one focal point.
- The jets of water from a fountain also follow a parabolic shape; this is helpful in predicting the paths of large quantities of water.
- The parabola is a useful shape in engineering. When you pass bridges, have a look at their shape and see how many of these are parabolic.

FF **You will learn more about parabolas in Chapter 18.**

Worked example 2.5

Q. Look carefully at the following equations. Identify the quadratic equations and rearrange them into the form $ax^2 + bx + c = 0$.

(a) $x^2 = 5 - 2x$ (b) $3 + x^3 = 5x$ (c) $3 - 2x = \frac{4}{x^2}$

(d) $4 + x(x + 2) = 0$ (e) $x^2 = 7$ (f) $11 = x^2 + 2x^{-1}$

A. (a) $x^2 = 5 - 2x$ is a quadratic equation.

Rearranging gives $x^2 + 2x - 5 = 0$

> Add $2x$ to both sides, and subtract 5 from both sides.

(b) $3 + x^3 = 5x$ is not a quadratic equation because it contains an x^3 term.

(c) $3 - 2x = \frac{4}{x^2}$ is not a quadratic equation because it contains x^{-2} rather than x^2.

> The x^2 term is dividing the number 4, so we actually have $\frac{1}{x^2}$ or x^{-2}, not x^2. (See Learning links '*1B*' on page 24 if you need a reminder of negative indices.)

(d) $4 + x(x + 2) = 0$ is a quadratic equation.

Rearranging gives:

$4 + x^2 + 2x = 0$

$x^2 + 2x + 4 = 0$

> Expanding the brackets gives you an x^2 term.

This equation contains the reciprocal of $x(\frac{1}{x})$ so to write it in the general form of a quadratic equation, where we have bx, we would need to multiply through by x. Doing so creates the following equation: $11x = x^3 + 2$, which is not a quadratic because the highest power of x is 3, not 2.

continued . . .

(e) $x^2 = 7$ is a quadratic equation.

Rearranging gives $x^2 - 7 = 0$

(f) $11x = x^2 + 2x^{-1}$ is not a quadratic equation, it is cubic

$11x = x^2 + 2\frac{1}{x}$

Multiplying each side by x:

$11x^2 = x^3 + 2$

$x^3 - 11x^2 + 2$

Exercise 2.6

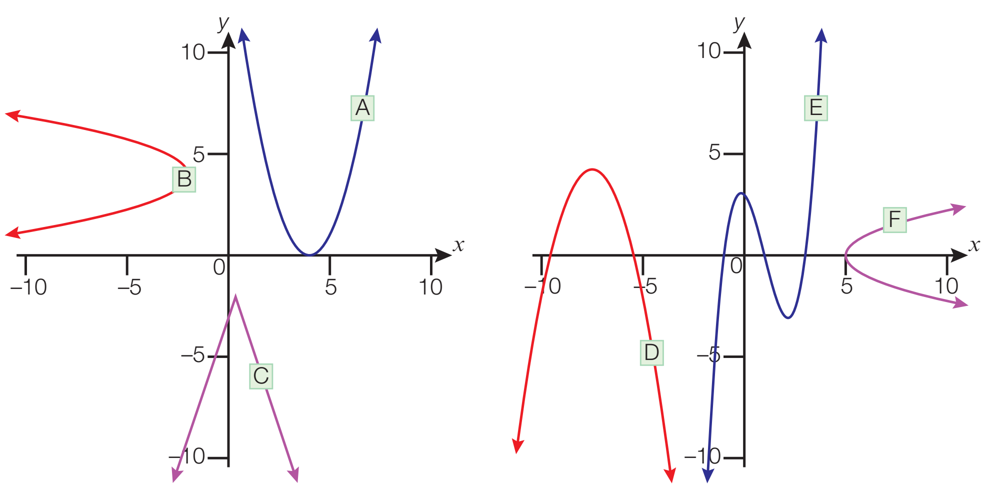
1. Determine which of the following are quadratic equations.

(a) $5x^2 = 7$ (b) $4 - x^2 = 0$ (c) $x^2 + x^3 - 7x = 5$

(d) $x^2 - 11x + 28 = 0$ (e) $\frac{5}{x^2} + 3x - 2 = 0$ (f) $10 - x - 9x^2 = 0$

(g) $3x^2 + \frac{5}{x} - 6 = 0$ (h) $0 = 1 + x + x^2$ (i) $-x^2 - 4(x + 1) = 0$

2. Six graphs are shown below. Indicate which of them represent quadratic functions.



3. The equations of some quadratic functions are given below. Indicate whether the parabola will have a minimum or maximum point.

(a) $y = 2x - x^2$ (b) $y = x^2 - 10x + 2$

(c) $y = 2x^2 + 3x - 5$ (d) $y = 56 + x - 7x^2$

4. Rearrange the following quadratic equations into the general form $ax^2 + bx + c = 0$.

(a) $x^2 = -x$ (b) $x^2 - 2x = 3$ (c) $4x^2 = 4 - x$

(d) $6 - x^2 = -5x$ (e) $x^2 + 5x - 8 = 7$ (f) $8 - 6x = 15 - x^2$

Solving quadratic equations

hint

Remember that the **solution**(s) of an equation are the numerical value(s) for the variables that make the equation true.

To solve a quadratic equation, you need to find the value(s) of the unknown variable that makes the equation $ax^2 + bx + c = 0$, true. In other words, the value(s) of x that make $ax^2 + bx + c$ equal zero.

You might have noticed that we talked about the 'value(s)' of the unknown variable. The '(s)' signifies that there can be more than one value for the variable. For quadratic equations, there are actually three possible outcomes of solving the equation:

- no solution
- one solution
- two solutions.

When the equation has two solutions, this means that there are **two** values of the variable that will make the equation true. But be careful, you do **not** substitute both values into the equation at the same time! You substitute one **or** the other solution, and either one will be correct.

For example, the solutions to the quadratic equation $x^2 - 9x + 20 = 0$ are $x = 4$ and $x = 5$. It is wrong to substitute both 4 and 5 into the equation at the same time: $4^2 - (9 \times 5) + 20 \neq 0$.

Instead, you should substitute $x = 4$ into the equation to check it works, and then substitute in $x = 5$:

when $x = 4$, $(4 \times 4) - (9 \times 4) + 20 = 16 - 36 + 20 = 0$

and

when $x = 5$, $(5 \times 5) - (9 \times 5) + 20 = 25 - 45 + 20 = 0$

Both $x = 4$ and $x = 5$ are solutions.

Different maths books may use different names to refer to the solutions of quadratic equations. They are sometimes called 'zeros' or 'the **roots** of an equation' rather than 'solutions'.

Like the other types of equation we have met in this chapter, a quadratic equation can be solved using algebra, by drawing a graph or using your GDC. The GDC makes it much quicker to solve the equation. However, it is a good idea to also know how to solve quadratic equations using the more traditional methods of algebra or by drawing a graph, so that you can use one method to find the answer and another to check the answer.

exam tip

It is a good idea to learn both algebraic and graphical methods on your GDC and become comfortable with using them. In examinations, questions can be set that test your knowledge of either.

For more about how to use the more traditional methods to solve quadratic equations, see Learning links 2F and 2G on pages 58–59.

There are two different ways you can use your GDC to solve quadratic equations. You can plot a graph and see where the graph crosses the x-axis, or you can use an equation solver.

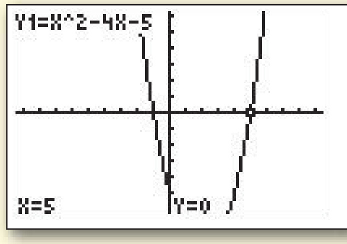
Worked example 2.6

Q. Solve the equation $x^2 - 4x - 5 = 0$, where the solutions are integers.

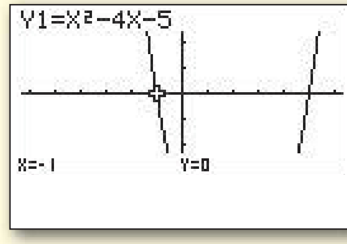
Use your GDC to draw the graph; make sure you can see the whole curve. See section *'2.3(a) Solving quadratic equations using a graph'* on page 656 of the GDC chapter if you need a reminder of how to do this.

A.

TEXAS



CASIO



Write down the answer appropriately.

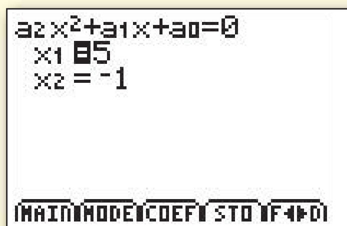
$x = -1$ or $x = 5$

hint

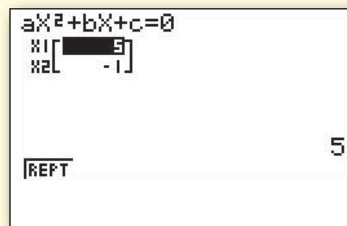
Remember that an integer is any whole number: positive, negative or zero.

Confirm your results using the equation solver on your GDC. See section *'2.3(b) Solving quadratic equations using an equation solver'* on page 657 of the GDC chapter if you need a reminder of how to do this.

TEXAS



CASIO



Answer is confirmed.

$x = -1$ or $x = 5$

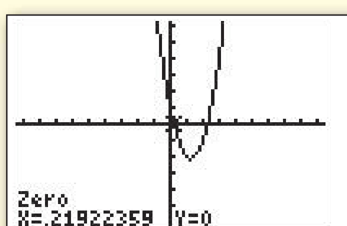
Worked example 2.7

Q. Solve the equation $2x^2 - 5x + 1 = 0$, where the solutions are not integers.

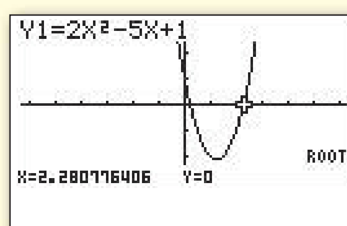
Use your GDC to draw the graph as in Worked example 2.6.

A.

TEXAS



CASIO



continued . . .

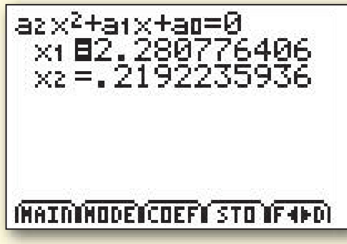
Write down the answers to three significant figures.

$x = 0.219$ (3 s.f.)

$x = 2.28$ (3 s.f.)

Confirm your results using the equation solver on your GDC as in Worked example 2.6.

TEXAS



CASIO

Answer is confirmed.

$x = 0.219$ (3 s.f.)

$x = 2.28$ (3 s.f.)

Learning links

2F Solving quadratic equations using algebra

When the equation is given in the form $(x - p)(x - q) = 0$

Suppose that you are asked to solve $(x - 5)(x + 1) = 0$.

One of the brackets, $(x - p)$ or $(x - q)$, must equal zero because only multiplying by zero will give zero! Write an equation making each bracket equal to zero to work out the possible values of x.

$5 - 5 = 0$

So, if $(x - 5) = 0$ then $x = 5$

$-1 + 1 = 0$

Or if $(x + 1) = 0$, then $x = -1$

Hence the solutions to this equation are $x = 5$ or $x = -1$.

When the equation is given in the form $ax^2 + bx + c = 0$ and $a = 1$

Suppose that you are asked to solve $x^2 - 2x - 3 = 0$.

This equation needs to be factorised before it can be solved. To **factorise** means to put into brackets so when $a = 1$, factorising means that $ax^2 + bx + c$ becomes $a(x + p)(x + q)$. In this example, you need to think about what value of p and what value of q would add together to give you -2 (the number 'b' in front of x) and multiply together to give you -3 (the **constant** 'c')

→

continued . . .

In this case, $p = -3$ and $q = 1$ would work, because

$-3 + 1 = -2$

$-3 \times 1 = -3$

So we can rewrite $x^2 - 2x - 3 = 0$ as

$(x - 3)(x + 1) = 0$

and then use the process shown in the previous example to find that

$x = 3$ or $x = -1$

exam tip

Don't worry if you find it difficult to factorise. You can always use one of the other methods to solve the equation.

Be careful when factorising because the relationship is more complicated than demonstrated here when $a \neq 1$.

2G Solving quadratic equations using a graph

Suppose we wanted to solve the equation $-2x^2 - 3x = -5$.

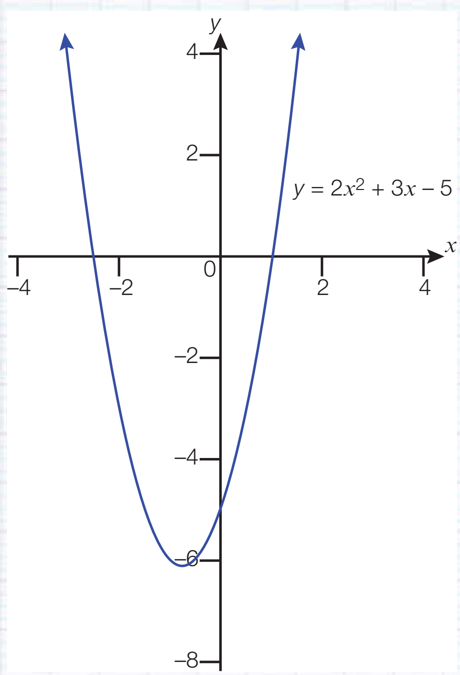
1. First make sure that the terms are arranged in the general form $ax^2 + bx + c = 0$.

   $2x^2 + 3x - 5 = 0$

Add $2x^2 + 3x$ to both sides of the original equation.

2. Plot the graph of $y = ax^2 + bx + c$

   $y = 2x^2 + 3x - 5$



3. You know that the solution(s) to the equation $ax^2 + bx + c = 0$ is the value of x that makes it true; the equation is true when $y = 0$. On any graph, $y = 0$ where the graph crosses the x-axis; the x-values at these points are the solutions. This means that a quadratic equation could have the following number of solutions:

- **zero**, if the graph does not meet the x-axis at all
- **one**, if the graph meets the x-axis at only one place
- **two**, if the graph crosses the x-axis at two points.

In the example, the graph cuts the x-axis at the two points (–2.5, 0) and (1, 0).

This means that the quadratic equation $2x^2 + 3x - 5 = 0$ has two solutions:

$x = -2.5$ or $x = 1$

Exercise 2.7

1. Solve the following quadratic equations using your GDC.

(a) $3x^2 + x - 4 = 0$ (b) $7x^2 - 18x + 8 = 0$

(c) $-6x^2 - 11x + 10 = 0$ (d) $-0.5x^2 + 4.8x + 6 = 0$

(e) $\frac{1}{3}x^2 - 5x + 1 = 0$ (f) $-\frac{5}{7}x^2 + \frac{2}{3}x + 1 = 0$

2. Rearrange these quadratic equations to the general form, and then solve them using your GDC.

(a) $2x^2 + 3x = 2$ (b) $3x^2 = 9 - 11x$ (c) $13 = x^2 + 7x$

(d) $3 - 6x^2 = 7x$ (e) $x(4x - 5) = 8$ (f) $4 - x(9x + 1) = 0$

3. Solve the following quadratic equations.

(a) $1.5x^2 - 2.3x - 4.7 = 0$ (b) $8x^2 + 13x = 11$

(c) $x(2x - 7) = 5$ (d) $4.1x^2 - 17x + 3.2 = 0$

(e) $x + 2 = \frac{1}{x}$ (f) $x = \frac{6}{3x + 1}$

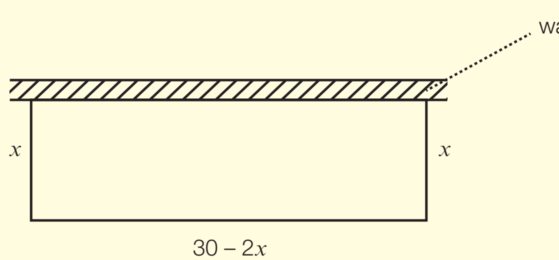
Solving practical problems involving quadratic equations

At the beginning of this chapter, you saw an example of the type of problem considered by Egyptian engineers thousands of years ago. Quadratic equations are still very useful in solving practical problems you might encounter in everyday life. For example, you can use quadratic equations to find out the area of a rectangular or circular space for a given perimeter, or estimate the time it takes for a stone to fall to the bottom of a well.

Worked example 2.8

Q. A farmer has 30 m of fencing to make a safe rectangular enclosure (a 'pen') for his lambs. He can use a wall to make one side, and the fencing to make the other three sides. He would like the area of the enclosure to be 108 m². Find the length and width of this pen.

A.



It is helpful to sketch a diagram. We know that there is 30 m of fencing for three sides because the fourth side is the wall. The pen is a rectangle, so we know that one side will be the same length as the wall and the other two sides will be the same length as each other. If we label the shorter sides x, then we know that the fencing needed for the two shorter edges is $2x$. The rest of the fence can be used for the edge opposite the wall. We can write an expression for this length:

$30 - 2x$.

continued . . .

Substitute the variables for the sides of the rectangle into the formula for the area of a rectangle.

Area of rectangle = length × width

Area = $x(30 - 2x)$

We know the area must equal 108 m^2 so we can write an equation to solve.

$x(30 - 2x) = 108$

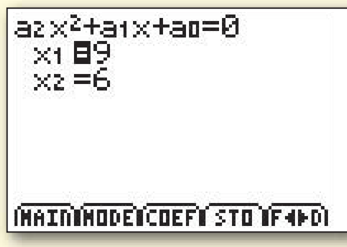
If you multiply out the brackets and rearrange, you will see that this is a quadratic equation.

$30x - 2x^2 = 108$

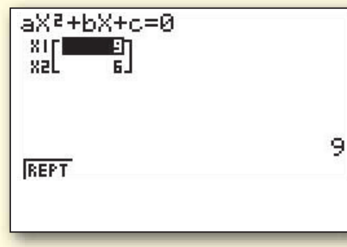
$-2x^2 + 30x - 108 = 0$

Use your GDC to solve.

TEXAS



CASIO



Two solutions are possible.

$x = 9$ or $x = 6$

Remember to find the length of the other side, $30 - 2x$, for each value of x.

If $x = 9$, then $30 - 2x = 30 - 2 \times 9 = 12$ m

If $x = 6$, then $30 - 2x = 30 - 2 \times 6 = 18$ m

If $x = 9$, the dimensions are 9 m by 12 m.

If $x = 6$, the dimensions are 6 m by 18 m.

Exercise 2.8

1. The product of two consecutive positive integers is 306. If the smaller number is n, write an equation in the form $an^2 + bn + c = 0$ to represent the product of the numbers. Hence solve the equation to find the two numbers.

2. The length of a rectangle is 7 cm more than its width. It has a diagonal of length 13 cm.

   (a) Write down a quadratic equation in x, using the information you have been given. (*Hint*: You may use Pythagoras' theorem.)

   (b) Solve this equation to find the dimensions of the rectangle.

3. The lengths of the parallel sides of a trapezium are x cm and $(x + 6)$ cm. The distance between the parallel sides is $(x - 2)$ cm, and the area of the trapezium is $150\,\text{cm}^2$.

   (a) Form a quadratic equation in terms of x for the area of the trapezium.

   (b) Solve the equation to find x.

   (c) Hence find the dimensions of the trapezium.

4. A stone is thrown directly upwards. The height of the stone above ground level, h metres, after t seconds is given by the formula:

   $$h = 13 + 8t - 1.9t^2$$

   (a) Calculate the time it takes for the stone to reach a height of 20 m above the ground.

   (b) How long does it take before the stone hits the ground?

Summary

You should know:

- how to recognise the following types of equation:
  - linear equations have one unknown and the general form $y = mx + c$ or $ax + by + c = 0$
  - pairs of linear equations contain two unknowns, e.g. x and y
  - quadratic equations have one unknown and the general form $ax^2 + bx + c = 0$ where $a \neq 0$ (but b and c can be 0)
- that rearranging equations is required in order to solve them, even if using your GDC
- how to use your GDC to solve:
  - linear equations with one variable
  - pairs of linear equations with two variables
  - quadratic equations

  by

  - drawing a graph
  - using the equation solver

  and be confident using both methods.

Mixed examination practice

Exam-style questions

1. Solve the equation $2(x - 3) + 5x = 36$.

2. Solve the following pairs of simultaneous equations:

(a) $x + 2y = 3$ and $4x + 3y = 2$

(b) $3x + 7y - 20 = 0$ and $11x - 8y + 5 = 0$

(c) $5x + 8y + 2 = 0$ and $6x - y + 19 = 0$

3. Solve the following quadratic equations:

(a) $(7x - 2)(3x - 1) = 0$ (b) $x^2 - 3x - 9 = 0$

(c) $x^2 + 2x = 34$ (d) $x^2 = 7x + 15$ (e) $7.5x^2 - 6x = 9.8$

4. Igor and Irishka are returning from a trip to India. They have bought presents for their friends and family.

Igor bought 4 bracelets and 3 pendants for 5529 INR (rupees).

Irishka bought 2 bracelets and 5 pendants for 6751 INR.

(a) Form two simultaneous equations, using the information given.

(b) Solve the pair of equations to find the price of each bracelet and each pendant.

5. The line with equation $y = mx + c$ passes through the points with coordinates $(1, -5)$ and $(4, 4)$.

(a) Write down two equations to represent the fact that the line passes through the two points given above.

(b) Solve the equations to find the values of m and c.

(c) Does the point $(-4, 11)$ lie on this line?

6. Safe Power supplies electricity to the Ahmed household. The bill for a three-month period consists of two parts: a fixed (standing) charge and a usage charge. The total charge per quarter, C, in pounds can be represented as $C = a + bn$, where a is the fixed charge per quarter, b is the unit charge (per kWh used), and n is the number of units (kWh) used in that quarter.

Last quarter the Ahmeds used 820 units of electricity and paid £106.24.

In the previous quarter they used 650 units and paid £85.84.

(a) Write down two equations in the variables a and b.

(b) Find the values of a and b.

(c) How much can the Ahmeds expect to pay next quarter if their estimated electricity usage is 745 units in the next three months?

7. A rectangular field has length 18 m longer than its width. If the area of the field is 760 m^2, calculate the dimensions of the field.

8. A ball is projected vertically into the air from ground level. After t seconds, the ball reaches a height of h metres. The equation for the flight of the ball can be expressed as:

$$h = 11t - 2.3t^2$$

(a) How long does it take the ball to reach a height of 10 metres?

(b) Calculate the time it takes for the ball to return to ground level.

Past paper questions

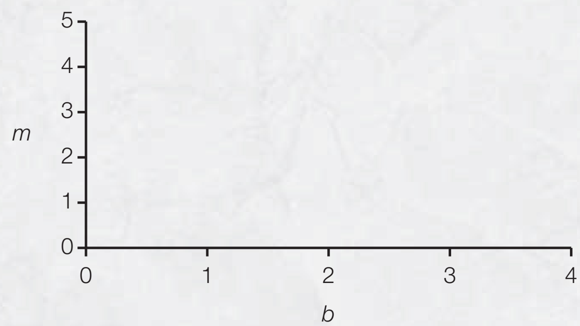
1. A store sells bread and milk. On Tuesday, 8 loaves of bread and 5 litres of milk were sold for \$21.40. On Thursday, 6 loaves of bread and 9 litres of milk were sold for \$23.40.

If b = the price of a loaf of bread and m = the price of one litre of milk, Tuesday's sales can be written as $8b + 5m = 21.40$.

(a) Using simplest terms, write an equation in b and m for Thursday's sales.

(b) Find b and m.

(c) Draw a sketch, in the space provided, to show how the prices can be found graphically.



[May 2007, Paper 1, Question 12] (© *IB Organization 2007*) *[6 marks]*

2. *It is not necessary to use graph paper for this question.*

(a) Sketch the curve of the function $f(x) = x^3 - 2x^2 + x - 3$ for values of x from -2 to 4, giving the intercepts with both axes.

[3 marks]

(b) On the same diagram, sketch the line $y = 7 - 2x$ and find the coordinates of the point of intersection of the line with the curve.

[3 marks]

hint

$f(x)$ is function notation and you will cover this in Chapter 17; for this question, just think of it as '$y =$ '.

[Nov 2007, Paper 2, Question 1(ii) (a),(b)] (© *IB Organization 2007*)