Chapter 3 Arithmetic and geometric sequences and series

A mosaic from within the Sultan Qaboos Grand Mosque, Mustat, Oman.

In this chapter you will learn:

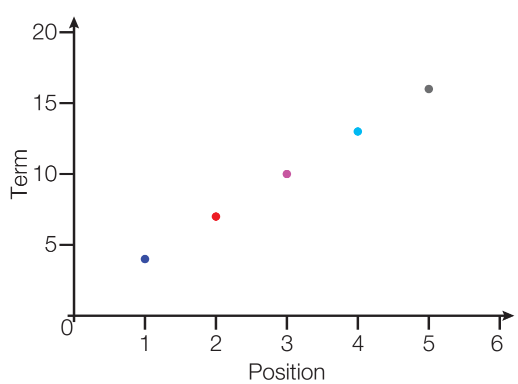
- about arithmetic sequences and series, and their applications
- about geometric sequences and series, and their applications.

Patterns are everywhere. Some people recognise them most easily in art, others in music or poetry. There are also patterns in numbers that can help you to understand mathematical ideas better.

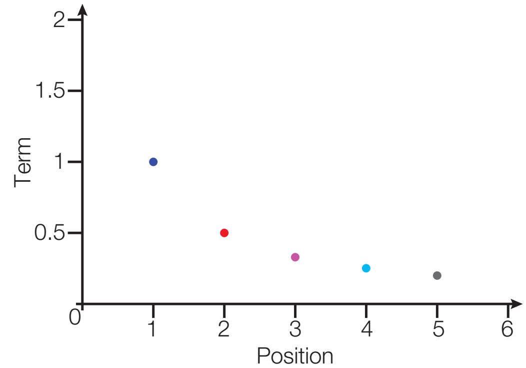
3.1 Arithmetic sequences

A **sequence** is an ordered list of numbers. In some sequences, the numbers have a regular pattern to them. Look at the sequences below. Each has been represented as a graph that plots the **position** of each **term** in the sequence against the term's **value**. If you look at either the list of numbers or the graph, you will see that there is a pattern in each sequence, and you can use this pattern to predict the next few numbers.

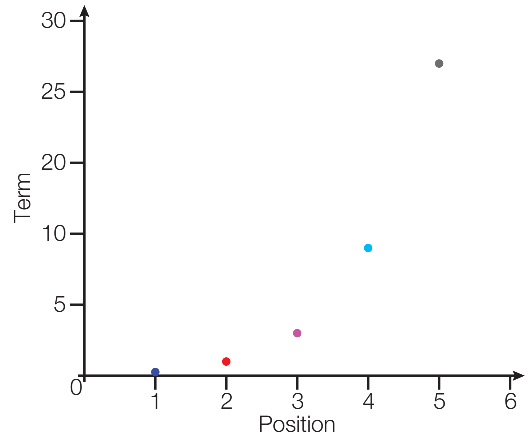
1. 4, 7, 10, 13, 16, …



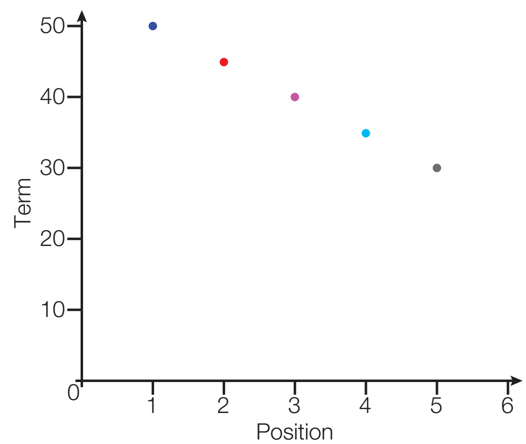
2. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$



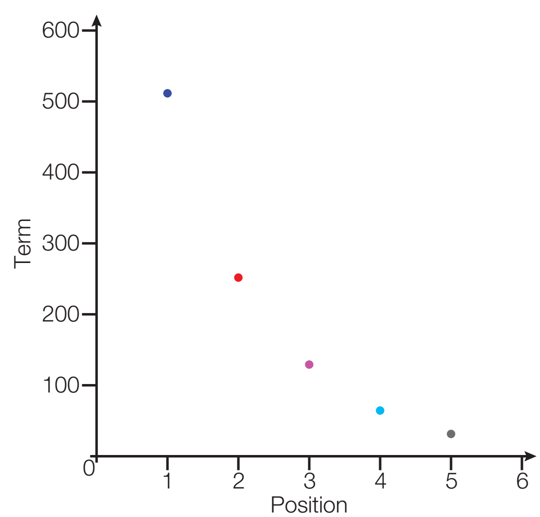
3. $\frac{1}{3}, 1, 3, 9, 27, \ldots$



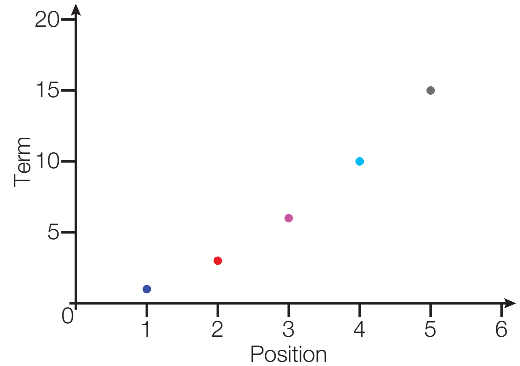
4. 50, 45, 40, 35, 30, …



5. 512, 256, 128, 64, 32, …



6. 1, 3, 6, 10, 15, …



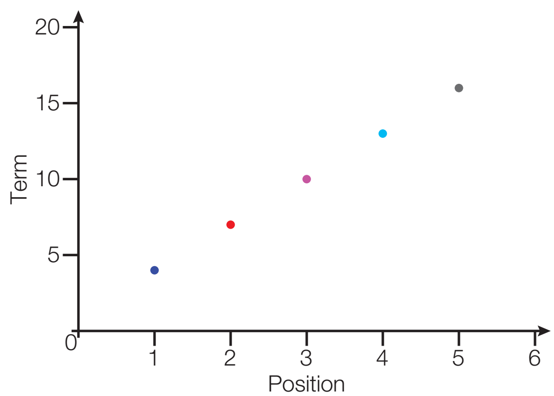
Look at either the number pattern or the graph. To get the next term, in which sequence would you:

- subtract 5?
- multiply by 3?
- divide by 2?
- add 3?

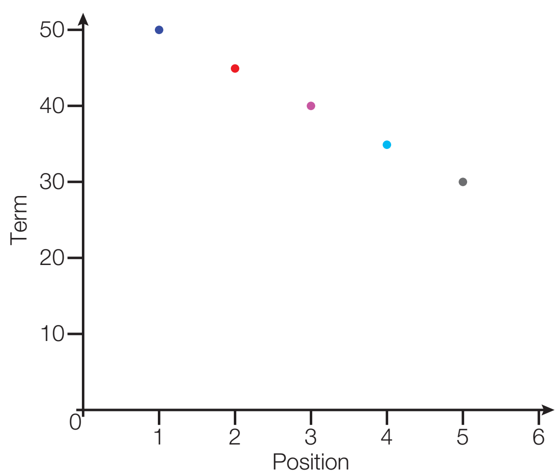
Two of the sequences are more complicated. Which are they? Can you find a pattern and describe how each of these sequences is built up?

Look again at the graphs of sequence 1 and sequence 4:

1.



4.



Notice that for each sequence, the plotted points lie along a straight line; the line is increasing in the case of sequence 1 and decreasing in the case of sequence 4. A sequence like 1 or 4 above is called an **arithmetic sequence** or **arithmetic progression**: the number pattern starts at a particular value and then increases, or decreases, by the same amount from each term to the next. This fixed **difference** between consecutive terms is called the **common difference** of the arithmetic sequence.

Look at sequence 1:

The sequence starts at 4, and increases by 3 for each subsequent term.

$$4 \xrightarrow{+3} 7 \xrightarrow{+3} 10 \xrightarrow{+3} 13 \xrightarrow{+3} 16$$

$7 - 4 = 10 - 7 = 13 - 10 = 16 - 13 = 3$, so **3** is the common difference of this sequence.

Now look at sequence 4:

The sequence starts at 50, and decreases by 5 for each subsequent term.

$$50 \xrightarrow{-5} 45 \xrightarrow{-5} 40 \xrightarrow{-5} 35 \xrightarrow{-5} 30$$

$45 - 50 = 40 - 45 = 35 - 40 = 30 - 35 = -5$, so **−5** is the common difference for this sequence.

Exercise 3.1

1. Which of the following are arithmetic sequences?

(a) 1, 2, 4, 8, 16, …

(b) 2, −2, 4, −4, 8, …

(c) 2, 9, 16, 23, 30, …

(d) 14, 8, 2, −4, −10, …

(e) 2, 3, 5, 8, 13, …

(f) 5, 12, 19, 26, 33, …

2. For each of the following arithmetic sequences, state the common difference and find the next three terms.

(a) 5, 8, 11, __, __, __, …

(b) 2, −1, −4, __, __, __, …

(c) 350, 317, 284, __, __, __, …

(d) 189, 210, 231, __, __, __, …

(e) 28.7, 32.9, 37.1, __, __, __, …

(f) $\frac{1}{2}, \frac{5}{4}, 2,$ __, __, __, …

(g) $\frac{4}{7}, \frac{26}{21}, 1\frac{19}{21}$

(h) $2x + 7, x - 2, -11,$ __, __, __, …

The *n*th term of an arithmetic sequence

We may want to know the value of a particular term in an arithmetic sequence, for example, the tenth term. We can use algebra to represent the terms of a sequence using the letter u and the position of each term in the sequence by a subscript number:

Position	1	2	3	4	…	$n-1$	n
Term	u_1	u_2	u_3	u_4	…	u_{n-1}	u_n

u_1 = first term, u_2 = second term, u_3 = third term and so on. A general term in a sequence is called the nth term. A term that is at position n in the sequence would be represented by u_n, and the term before u_n would be represented by u_{n-1}.

There is a general formula to calculate the nth term of an arithmetic sequence:

$u_n = u_1 + (n - 1)d$, where d is the common difference.

The general formula allows you to calculate the value of any term in an arithmetic sequence, so long as you know the value of the starting term (u_1) and the common difference (d).

Worked example 3.1

Q. (a) If $u_1 = 5$ and $d = 4$, find u_8 and u_{14}.

(b) For the sequence 8, 7.5, 7, 6.5, … calculate u_9 and u_{50}.

A. (a) Substitute the given values of u_1, d and n into the formula for the general term u_n. First, let $n = 8$.

$u_n = u_1 + (n - 1)d$

$u_8 = 5 + (8 - 1) \times 4 = 5 + 7 \times 4 = 33$

Substitute $n = 14$ into the formula.

$u_{14} = 5 + (14 - 1) \times 4 = 5 + 13 \times 4 = 57$

(b) Identify the values of u_1 and d for this sequence.

In this sequence, $u_1 = 8$ and $d = -0.5$.

Substitute $n = 9$ into the formula for the general term u_n.

$u_9 = 8 + (9 - 1) \times (-0.5) = 8 - 4 = 4$

Substitute $n = 50$ into the formula.

$u_{50} = 8 + (49 - 1) \times (-0.5) = 8 - 24 = -16$

Learning links

3A Deriving the formula for the *n*th term of an arithmetic sequence

The pattern in an arithmetic sequence can be used to analyse its structure and to find the general rule that allows you to calculate the value of any term in the sequence.

Look the sequence 5, 9, 13, 17, 21, …

The first term is 5, and the common difference is +4.

Write out the terms in a way that helps you to spot a pattern:

		Or	What is done to u_1
The first term	$u_1 = 5$	$u_1 = 5$	
The second term	$u_2 = 5 + 4 = 9$	$u_2 = 5 + 4$	Adding one 4
The third term	$u_3 = 9 + 4 = 13$	$u_3 = 5 + 4 + 4$	Adding two 4s
The fourth term	$u_4 = 13 + 4 = 17$	$u_4 = 5 + 4 + 4 + 4$	Adding three 4s

Notice that to get u_3, you have to add 4 to the first term and then add 4 again, i.e. you need to add 2 lots of 4 to the first term.

To get u_4 you have to add 3 lots of 4 to the first term.

The pattern shows that the number of times you need to add 4 is one fewer than the term number:

$$u_4 = u_1 + (4 - 1) \times 4$$

Now replace the numbers with letters: *n* for the position of the term in the sequence and *d* for the common difference. Then according to the pattern, to find any term (at the *n*th position) in the sequence, you:

take the first term, u_1, and then add the common difference, *d*, one time fewer than the term number that you need.

This is easier to write as a formula:

$$u_n = u_1 + (n - 1) \times d$$

Let us check using a different sequence.

Consider the sequence 100, 95, 90, 85, 80, …

The common difference is –5. Substitute the values of *n* and *d* into the formula and see if you get the correct values for the terms of the sequence:

$$u_1 = 100$$
$$u_2 = 100 + (2 - 1) \times (-5) = 100 - 5 = 95$$
$$u_3 = 100 + (3 - 1) \times (-5) = 100 - 10 = 90$$
$$u_4 = 100 + (4 - 1) \times (-5) = 100 - 15 = 85$$

It works!

This is true for all arithmetic sequences.

Exercise 3.2

1. In each of the following sequences, you are given the first term u_1 and the common difference d. Find the requested terms of the sequence using the formula for the nth term.

   (a) $u_1 = 7$, $d = 6$; find the 19th and 27th terms.

   (b) $u_1 = 36$, $d = 21$; find the 20th and 40th terms.

   (c) $u_1 = 84$, $d = -13$; find the 3rd and 17th terms.

   (d) $u_1 = -23$, $d = 11$; find the 16th and 34th terms.

   (e) $u_1 = -156$, $d = 29$; find the 10th and 18th terms.

   (f) $u_1 = 1080$, $d = -15.6$; find the 8th and 21st terms.

   (g) $u_1 = 268$, $d = -16$; find the 41st and 69th terms.

   (h) $u_1 = 59.4$, $d = 12.3$; find the 31st and 55th terms.

   (i) $u_1 = \frac{3}{7}$, $d = \frac{1}{5}$; find the 18th and 27th terms.

2. For each of the following arithmetic sequences, calculate the terms indicated.

   (a) 2, 5, 8, … ; 7th and 11th terms

   (b) 16, 23, 30, … ; 20th and 31st terms

   (c) 35, 39, 43, … ; 9th and 40th terms

   (d) 0, −4, −8, …. ; 23rd and 30th terms

   (e) 2, −7, −16, … ; 11th and 29th terms

   (f) 120, 77, 34, … ; 10th and 27th terms

   (g) 0.62, 0.79, 0.96, … ; 18th and 35th terms

   (h) $\frac{5}{9}, \frac{29}{36}, \frac{19}{18}, \ldots$; 7th and 21st terms

   (i) $5x + 2, 6x + 7, 7x + 12, \ldots$; 13th and 20th terms

Using the formula to find values other than u_n

The general formula can be used in a number of different ways depending on what you know about the sequence and what you want to find out.

If you know the value of the first term and the value of any other term in the sequence, you can work out the common difference, even if you do not have all the terms of the sequence.

If you know at least three consecutive terms (terms next to each other) in a sequence then you can work out the common difference easily. If you also know the last term of the sequence, you can work out how many terms there are in the sequence.

Worked example 3.2

There are two ways of answering this question.

Q. The fourth term of an arithmetic sequence is 17. The first term is 5.

Find the common difference.

Look at the way the sequence is set up.

A. Method 1:

5 _?_ _?_ 17, _?_, _?_...

To get the 4th term, we add 3 lots of the common difference to the first term. So you know that between 5 and 17, d was added three times. Write this as an equation.

$$3d = 17 - 5$$

Solve for d.

$$3d = 12$$
$$d = 4$$

Use the general formula for u_n and substitute in the values you know.

Method 2:

$$u_n = u_1 + (n-1)d$$

$u_1 = 5$ and $u_4 = 17$, so $17 = 5 + (4-1) \times d$

Solve for d.

$$17 = 5 + 3d$$
$$17 - 5 = 3d$$
$$12 = 3d$$
$$d = 4$$

Worked example 3.3

Q. Find the common difference and the number of terms in the sequence 49, 43, 37, … , 1.

The common difference is easy to work out first.

A. $d = 43 - 49 = -6$

There are several different ways to calculate the number of terms in the sequence.

Method 1:

Use the formula for the general term u_n.

Substitute the information that you know into the formula.

$u_n = u_1 + (n-1)d$

$1 = 49 + (n-1) \times (-6)$

This is a linear equation in the variable n.

Solve for n.

So 1 is the 9th term of the sequence.

$1 = 49 - 6(n-1)$

$6(n-1) = 49 - 1$

$6(n-1) = 48$

$n - 1 = 8$

$n = 9$

RR ***Solving linear equations was covered in Chapter 2.***

There are 9 terms in the sequence.

Use the recursion mode of your GDC to enter the common difference repeatedly.

See section '*3.1 (a) Finding the number of terms in an arithmetic sequence using the recursion mode to enter the common difference repeatedly*' on page 658 of the GDC chapter for a reminder, if you need to.

Method 2:

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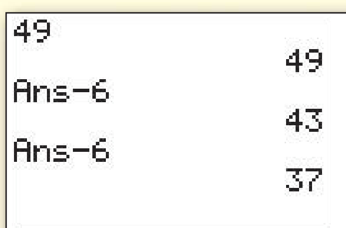

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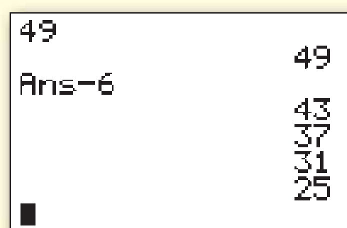

1 is the 9th term.

So there are 9 terms in the sequence.

continued . . .

Use the formula for a general term and an appropriate equation solver on your GDC.

Method 3:

Substitute known values into the formula.

$u_n = u_1 + (n - 1)d$

$1 = 49 + (n - 1) \times (-6)$

Rearrange the formula so that the equation equals zero.

$6(n - 1) - 48 = 0$

Use your GDC to solve. See section '2.1 (b) *Solving linear equations using an equation solver*' on page 653 of the GDC chapter if you need to.

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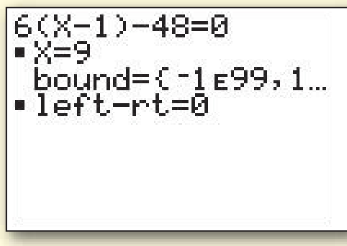

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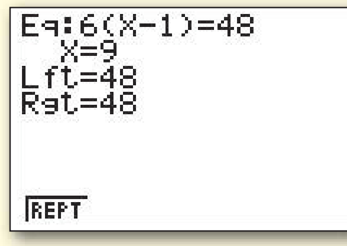

Write the answer down appropriately.

There are 9 terms in the sequence.

Sometimes you might know two terms in a sequence that are neither the first term nor consecutive terms. In these circumstances, it is a little more complicated to calculate the common difference, but it is still possible. You can use the formula for the general term u_n, and the terms that you *do* know, to write a pair of linear equations to solve.

Worked example 3.4

Q. If the third term of an arithmetic sequence is 12 and the eighth term is 27, find the first term and the common difference.

Substitute known values into the formula for the general term u_n; this gives a pair of linear equations with the unknowns u_1 and d.

A. $u_n = u_1 + (n - 1)d$

$u_3 = u_1 + (3 - 1)d$, so $12 = u_1 + 2d$

$u_8 = u_1 + (8 - 1)d$, so $27 = u_1 + 7d$

Now you have a pair of simultaneous equations with two unknown quantities to find.

$u_1 + 2d = 12$

$u_1 + 7d = 27$

Solve using **your** preferred method.

RR *In Chapter 2 you learned techniques for solving pairs of linear equations; you will need to use one of those techniques here.*

continued . . .

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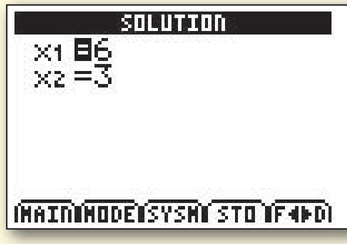

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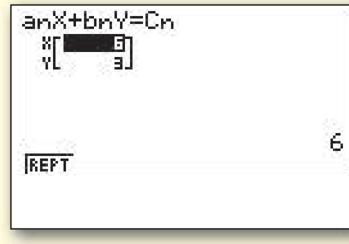

Write down the answer appropriately.

$u_1 = 6$ and $d = 3$

Exercise 3.3

1. In the following arithmetic sequences you are given the first term and one other term of the sequence. Find the common difference in each case.

(a) $u_1 = 7$ and $u_{18} = 58$ (b) $u_1 = 45$ and $u_{10} = 117$

(c) $u_1 = -17$ and $u_{22} = 214$ (d) $u_1 = 25.9$ and $u_{30} = -40.8$

(e) $u_1 = 87$ and $u_{25} = -240.6$ (f) $u_1 = -40$ and $u_{17} = 88$

(g) $u_1 = -135$ and $u_{18} = 307$ (h) $u_1 = 19.7$ and $u_{31} = -43.3$

(i) $u_1 = 66.1$ and $u_{50} = -27$ (j) $u_1 = 19.84$ and $u_{102} = 76.703$

2. Find the number of terms in each of the following arithmetic sequences. You are given the first three terms and the last term in each case.

(a) 5, 7, 9, … , 75 (b) 15, 18, 21, … , 93

(c) 64, 77, 90, … ,649 (d) −6, 7, 20, … ,488

(e) 49, 60, 71, … , 643 (f) 80.8, 75.9, 71, … , −404.3

(g) 37.95, 34.3, 30.65, … , −126.3

(h) 126.4, 117.95, 109.5, … , −498.9

(i) 167, 133, 99, … , −1363

(j) 1083, 1064, 1045, … , 0

3. Find the first term in each of the following arithmetic sequences. In each case you are given the common difference and one term in the sequence.

(a) $d = 5$ and $u_{12} = 67$
(b) $d = 17$ and $u_{14} = 240$
(c) $d = -8$ and $u_{51} = 0$
(d) $d = -23$ and $u_{27} = -400$
(e) $d = 9.75$ and $u_{24} = 280.25$
(f) $d = -6.9$ and $u_{14} = 98.3$
(g) $d = 54$ and $u_{40} = 4096$
(h) $d = 13.6$ and $u_{33} = 523.2$
(i) $d = -10.1$ and $u_{78} = -572.7$
(j) $d = \frac{2}{3}$ and $u_{24} = 75\frac{1}{2}$

4. The fifth term of an arithmetic sequence is 9 and the eleventh term is 45.

(a) Denoting the first term by u_1 and the common difference by d, write down two equations in u_1 and d that fit the given information.
(b) Solve the equations to find the values of u_1 and d.
(c) Hence find the fiftieth term of the sequence.

5. The fourth term of an arithmetic sequence is 118 and the seventh term is 172.

(a) Find the first term and the common difference.
(b) Calculate the twentieth term of the sequence.

6. The ninth term of an arithmetic sequence is 36 and the twenty-first term is −168.

(a) Find the first term and the common difference.
(b) Calculate the thirty-seventh term.

hint

The position of a term in a sequence has to be an integer.

7. In an arithmetic sequence the tenth term is −88.93 and the seventeenth term is −130.93.

(a) Determine the common difference and the first term.
(b) Find the fortieth term of the sequence.
(c) Is −178.52 a term in the sequence?

Solving practical problems involving arithmetic sequences

Patterns are seen in art, music and poetry. But patterns can also be found in many other contexts that may not be so obvious. For example, seating plans in theatres and sports arenas, or the growth of a child all display patterns.

You can apply what you have learned about arithmetic sequences to practical situations like these as well as many others.

Worked example 3.5

As the distance Maya runs increases by a fixed amount each week, her training pattern is an example of an arithmetic sequence with $u_1 = 5$ and $d = 3$. There are two ways you could answer the questions.

Q. Maya is training for a marathon. She builds up her fitness by running an extra 3 km each week. She runs 5 km the first week.

(a) How far will she run in the third week?

(b) In which week will she run more than 25 km?

Use the recursion mode on your GDC until there are three terms on the screen.

A. (a) Method 1:

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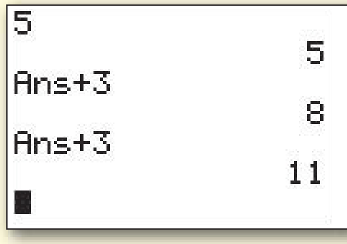

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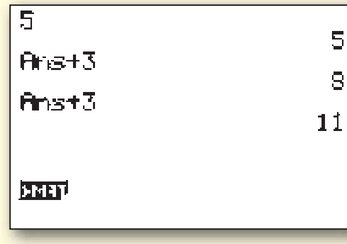

The third term is 11.

Maya runs 11 km in the third week.

Substitute known values into the formula for the general term: $u_1 = 5$, $d = 3$, and $n = 3$.

Method 2:

$u_n = u_1 + (n-1)d$

$u_3 = 5 + (3-1) \times 3 = 11$ km

Substitute $u_1 = 5$ and $d = 3$ into the formula for the general term. We are looking for the value of n for which $u_n > 25$. We need to solve the inequality $5 + 3(n-1) > 25$. There are several ways you can do this.

(b) $u_n = 5 + 3(n-1)$

$5 + 3(n-1) > 25$

Return to Chapter 2 for a reminder of how to solve linear equations using your GDC if you need to.

Method 1:

Use an equation solver on your GDC. The inequality sign (>) needs to be replaced by an = sign, so that we can treat it as a linear equation.

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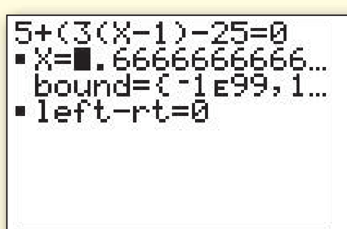

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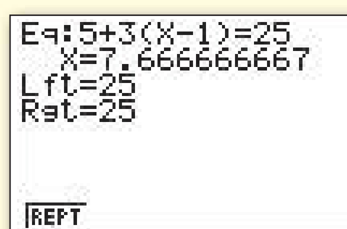

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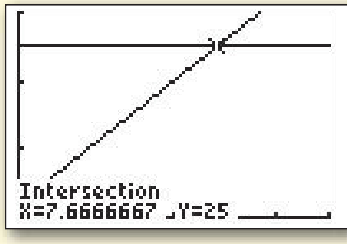
The answer given by the GDC is 7.67, but the question asks for 'which week', so take $n = 8$. (You can check that $n = 7$ gives 23 m, which is less than 25.)

Maya runs more than 25 km in the 8th week.

Method 2:

Use your GDC to draw the graph of the sequence $y_1 = 5 + 3(x - 1)$ and the line $y_2 = 25$. The point of intersection tells us the value of x for which the sequence equals 25.

TEXAS



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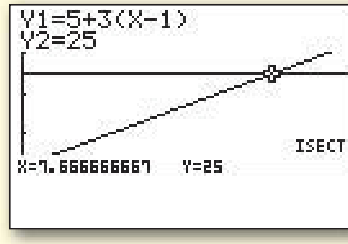


The GDC gives the x-coordinate of the intersection as 7.67. Round this up to 8 to get the value of n.

Maya runs more than 25 km in the 8th week.

hint

Note in Worked example 3.5 that n is a number of weeks and must be a natural number, so the answer should be rounded up to the next natural number.

Exercise 3.4

1. Jamie is collecting Pokemon cards. In the first month he collected 12 cards, and he plans to collect an additional 7 cards every month. The total number of cards in his collection each month forms an arithmetic sequence.

   (a) How many cards will Jamie have in the sixth month?

   (b) How long will it take Jamie to collect 96 cards?

2. Rosetta has bought a new Russian language phrase-book. She has decided to learn some new Russian words every week. In the first week she learned 10 new words. She learned 19 new words in the second week and 28 new words in the third week. The number of new Russian words Rosetta learns each week forms an arithmetic sequence.

   (a) How many new words will Rosetta learn in the eleventh week?

   (b) During which week will Rosetta learn 181 new words?

3. Sally has 30 weeks of training before her next sporting event. In the first week she trains for 45 minutes. The lengths of time she trains every week form an arithmetic sequence. Each week she trains four minutes longer than in the previous week.

(a) How long will Sally train in the fourteenth week?

(b) After which week will Sally be training longer than two hours?

(c) How long will she train in the final week before the sporting event?

4. Mr Mensah owns several cocoa plantations. Each year he plans to harvest 18 tonnes more cocoa beans than in the previous year. In the first year he harvested 42 tonnes, the following year 60 tonnes, the year after that 78 tonnes, and so on.

(a) How many tonnes of cocoa beans does Mr Mensah expect to harvest in the sixth year?

(b) In which year will Mr Mensah's harvest exceed 300 tonnes of cocoa beans?

5. Veejay works as a car salesman. His monthly commission forms an arithmetic sequence. In the tenth month he earned 2150 rupees. In the twenty-first month he earned 3800 rupees.

(a) How much commission did Veejay earn in the first month?

(b) In which month is his commission expected to exceed 6000 rupees?

3.2 Arithmetic series: the sum of an arithmetic sequence

If you add up the terms of an arithmetic sequence, the result is called an **arithmetic series**:

$S = u_1 + u_2 + u_3 + u_4 + u_5 + \ldots + u_{n-1} + u_n$

You can use arithmetic **series** to solve different problems.

Let us return to Maya training for a marathon (Worked example 3.5). She builds up her fitness by running an extra 3 km each week. She runs 5 km in the first week.

Maya now wishes to know the total distance that she has run in her first eight weeks of training. How can she find this out?

She could write down the distance run in each of the eight weeks and add these numbers up:

$S = 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 = 124\,\text{km}$

To do this, she first needs to work out each term in the sequence. This is easy when there are only eight terms, but if Maya wanted to know the total distance run in a year (52 weeks), it would be a long task to work out every term of the sequence and then add them all up!

There are much quicker ways of calculating the sum of an arithmetic series, using either algebra or your GDC.

Using algebra to calculate the sum of an arithmetic series

The sum of an arithmetic series with n terms is given by the formula

$$S_n = \frac{n}{2}(u_1 + u_n)$$

hint

This is the formula to use if you know the first term, the last term, and the total number of terms. It is most helpful when there are lots of terms in the sequence, because you don't need to work out every term in order to get the sum!

Maya could work out the total distance run in the first eight weeks of training as $S_8 = \frac{8}{2}(5+26) = 124$ km

Exercise 3.5

1. For each of the following arithmetic series, you are given the first term u_1 and the last term u_n. Find the sum of the first n terms in each series using the formula $S_n = \frac{n}{2}(u_1 + u_n)$.

(a) $u_1 = 7$ and $u_{20} = 64$
(b) $u_1 = 35$ and $u_{32} = 376$
(c) $u_1 = -80$ and $u_{29} = 32$
(d) $u_1 = 79$ and $u_{40} = -194$
(e) $u_1 = 46.1$ and $u_{24} = 225.5$
(f) $u_1 = -20$ and $u_{18} = -362$

A reminder of how to rearrange equations was provided in Learning links 2A on page 40 of Chapter 2.

2. For each of the following arithmetic series, you are given the last term and the sum of the first n terms. Use the formula $S_n = \frac{n}{2}(u_1 + u_n)$ to work out the first term of each series. (*Note*: you may have to rearrange the formula.)

	Last term (u_n)	Number of terms (n)	Sum of series
(a)	5	10	185
(b)	119	20	1430
(c)	158	14	1302
(d)	−160	24	−1632
(e)	−7	30	1095
(f)	0	32	4960

What if you do not know the last term in the series whose sum you want to find? You cannot use the formula. There is another formula for the sum of an arithmetic series. This formula uses the common difference d instead of the last term:

$$S_n = \frac{n}{2}[2u_1 + (n-1)d]$$

In the case of Maya's training (Worked example 3.5), the total distance she runs in the first eight weeks can be calculated using this formula as follows:

$$S_8 = \frac{8}{2}[2\times 5 + (8-1)\times 3]$$
$$= 4\times[10+21]$$
$$= 124 \text{ km}$$

Exercise 3.6

1. Use the formula $S_n = \frac{n}{2}[2u_1 + (n-1)d]$ to find the sum of each of the following arithmetic series.

(a) $u_1 = 14$, $d = 8$ and $n = 10$

(b) $u_1 = 33$, $d = 16$ and $n = 18$

(c) $u_1 = -5$, $d = 27$ and $n = 21$

(d) $u_1 = 30$, $d = -19$ and $n = 20$

(e) $u_1 = -28$, $d = 1.5$ and $n = 40$

(f) $u_1 = 14$, $d = 8$ and $n = 10$

(g) $u_1 = 53$, $d = -7$ and $n = 29$

(h) $u_1 = 80.52$, $d = -13.75$ and $n = 30$

2. Use the formula $S_n = \frac{n}{2}[2u_1 + (n-1)d]$ to find the sum of each of the following series. For each series you are given the first three terms and the number of terms in the sum.

(a) $8 + 15 + 23 + \ldots$; 12 terms

(b) $9 + 20 + 31 + \ldots$; 20 terms

(c) $56 + 70 + 84 + \ldots$; 26 terms

(d) $145 + 95 + 45 + \ldots$; 28 terms

(e) $35 + 18 + 1 + \ldots$; 15 terms

(f) $12.5 + 20 + 27.5 + \ldots$; 18 terms

(g) $6.75 + 5.5 + 4.25 + \ldots$; 30 terms

(h) $3.172 + 4.252 + 5.332 + \ldots$; 36 terms

Learning links

3B Deriving the formula $S_n = \frac{n}{2}(u_1 + u_n)$ for the sum of an arithmetic series

Using Maya's training example (Worked example 3.5), let us try to spot a pattern in the numbers by writing the sequence forwards and then backwards:

sequence written forwards

S	=	5	+	8	+	11	+	14	+	17	+	20	+	23	+	26
S	=	26	+	23	+	20	+	17	+	14	+	11	+	8	+	5

sequence written backwards

We can see pairs of numbers. Notice that each pair adds up to the same value (31).

Adding each pair of numbers together, we get

$(5 + 26) + (8 + 23) + (11 + 20) + \ldots + (26 + 5)$
$= 31 + 31 + 31 + \ldots + 31$

until we have 8 lots of 31. So the sum is

$8 \times 31 = 248$

continued . . .

Notice we have used the same sequence of numbers twice (forwards and backwards), so we need to divide by 2:

$S = 248 \div 2 = 124$

With the same reasoning as above but using letters instead of numbers, we have

sequence written forwards

$$S_n = \boxed{u_1} + \boxed{u_2} + \boxed{u_3} + \dots + \boxed{u_{n-2}} + \boxed{u_{n-1}} + \boxed{u_n}$$
$$S_n = \boxed{u_n} + \boxed{u_{n-1}} + \boxed{u_{n-2}} + \dots + \boxed{u_3} + \boxed{u_2} + \boxed{u_1}$$

sequence written backwards

We have created n pairs of numbers and each pair has the same total, which is equal to $(u_1 + u_n)$. So, adding the sum of the two series together we get n lots of $(u_1 + u_n)$:

$2S_n = n(u_1 + u_n)$

Therefore $S_n = \frac{n}{2}(u_1 + u_n)$.

3C Deriving the formula $S_n = \frac{n}{2}[2u_1 + (n-1)d]$ for the sum of an arithmetic series

Let's use Maya's sequence again. In the bottom row, we have replaced the numbers with letters: n for the position of the term in the sequence, and d for the common difference.

$S = 5$	$+ 8$	$+ 11$	$+ \dots$	$+ 23$	$+ 26$
$S = 5$	$+ (5 + 3)$	$+ (5 + 3 + 3)$	$+ \dots$	$+ (5 + 3 + 3 + 3 + 3 + 3 + 3)$	$+ (5 + 3 + 3 + 3 + 3 + 3 + 3 + 3)$
$S = u_1$	$+ (u_1 + d)$	$+ (u_1 + 2d)$	$+ \dots$	$+ (u_1 + (n - 2)d)$	$+ (u_1 + (n - 1)d)$

Writing the sequence forwards and backwards we get:

sequence written forwards

$$S_n = \boxed{u_1} + \boxed{(u_1 + d)} + \boxed{(u_1 + 2d) + \dots} + \boxed{(u_1 + (n-2)d)} + \boxed{(u_1 + (n-1)d)}$$
$$S_n = \boxed{(u_1 + (n-1)d)} + \boxed{(u_1 + (n-2)d)} + \boxed{(u_1 + (n-3)d) + \dots} + \boxed{(u_1 + d)} + \boxed{u_1}$$

sequence written backwards

If we add each pair together then there are n pairs of numbers, and each pair adds up to the same total: $2u_1 + (n - 1)d$. So we have n lots of $2u_1 + (n - 1)d$ altogether, and we get

$2S_n = n[2u_1 + (n - 1)d]$

Dividing by 2 then gives $S_n = \frac{n}{2}[2u_1 + (n-1)d]$.

Using your GDC to calculate the sum of an arithmetic series

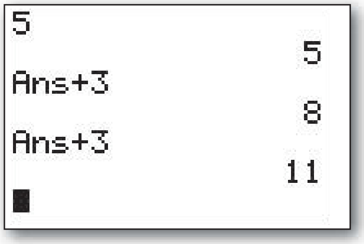
You can use your GDC to calculate the sum of a given number of terms in an arithmetic sequence. There are two main methods:

- using the GDC recursion mode
- using the 'sum' and 'seq' functions on your GDC.

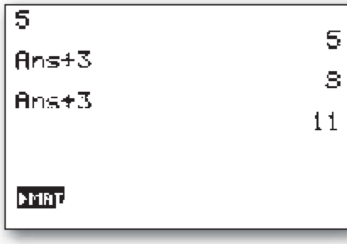
Using the recursion mode on your GDC

To calculate the total distance that Maya runs in eight weeks of training:

TEXAS



CASIO



Use your GDC in recursion mode to output each term in the sequence as far as the number of terms requested.

$5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 = 124$ km

Write down the numbers displayed on the right-hand side of the screen and add them together.

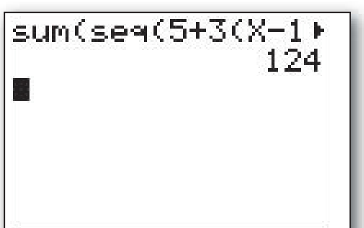
Using the 'sum' and 'seq' functions on your GDC

A function for summing the terms of a sequence is available on your GDC. See '*3.2 Finding the sum of an arithmetic sequence using the 'sum' and 'seq' functions*' on page 659 of the GDC chapter for a reminder if you need to.

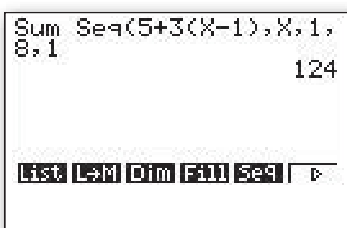
To calculate the total distance that Maya runs in eight weeks of training:

If $u_1 = 5$ and $d = 3$, $u_n = 5 + 3(n - 1)$

TEXAS



CASIO



First, put the known values into the formula for the nth term of an arithmetic sequence, $u_n = u_1 + (n - 1)d$.

Enter the right-hand side of the formula into your GDC and enter the required parameters.

$= 124$ km

Worked example 3.6

Q. Consider the sequence consisting of all the odd numbers from 1 to 99:

1, 3, 5, 7, 9, … , 97, 99

It is an arithmetic sequence with $u_1 = 1$ and $d = 2$.

(a) How many terms are there?

(b) What is the total if you add all the terms up?

Using algebra: first put the known values into the formula for an arithmetic sequence.

A. (a) $u_n = u_1 + (n-1)d$

$99 = 1 + 2(n-1)$

Rearrange and solve for n.

$98 = 2(n-1)$

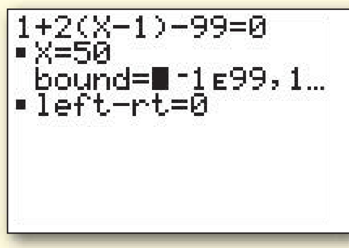
$49 = n-1$

$n = 50$

There are 50 terms in the sequence.

Check the answer using your GDC. See '*2.1 (b) Solving linear equations using an equation solver*' on page 653 if you need to.

TEXAS



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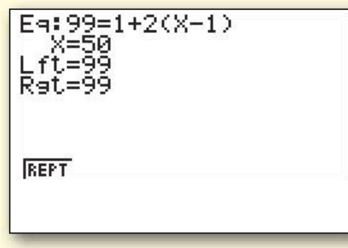


The answers match.

There are 50 terms in the sequence.

Using algebra: since we know the last term, we can substitute the known values into the formula $S_n = \frac{n}{2}(u_1 + u_n)$ for the sum of an arithmetic series.

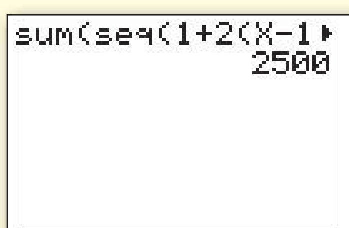
(b) $S_{50} = \frac{50}{2}(1+99)$

$= 25 \times 100$

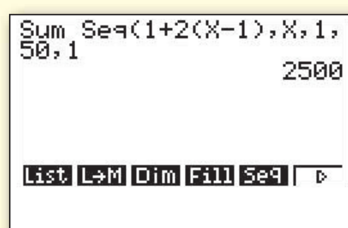
$= 2500$

Check your answer using the 'sum' and 'seq' functions on your GDC. See '*3.2 Finding the sum… using the 'sum' and 'seq' functions*' on page 659 if you need to.

TEXAS



CASIO



The sum of the sequence is 2500.

Exercise 3.7

Use algebra to answer the questions below, and check your answers with your GDC.

exam tip

Using a GDC can be quite complicated for this type of problem so you should also know how to calculate (or check) your answer using algebra.

1. For each of the following arithmetic series, find the sum of the specified number of terms.

(a) $7 + 15 + 23 + \ldots$ (24 terms)

(b) $38 + 51 + 64 + \ldots$ (16 terms)

(c) $150 + 127 + 104 + \ldots$ (40 terms)

(d) $4.97 + 8.19 + 11.41 + \ldots$ (36 terms)

(e) $\frac{3}{4} + \frac{19}{20} + 1\frac{3}{20} \ldots$ (15 terms)

2. For the following arithmetic series you are given the first three terms and the last term. In each case find the number of terms and the sum of the series.

(a) $14 + 27 + 40 + \ldots + 261$

(b) $86 + 115 + 144 + \ldots + 985$

(c) $7 + 8.35 + 9.7 + \ldots + 31.3$

(d) $93 + 76 + 59 + 42 + \ldots + (-400)$

(e) $12\frac{1}{2} + 15\frac{1}{4} + 18 + \cdots + 95$

3. Find the sum of each of the following series:

(a) The first 80 positive integers.

(b) All the even numbers between 23 and 243.

(c) All multiples of 3 between 2 and 298.

(d) The non-multiples of 7 between 1 and 99 inclusive.

(e) All common multiples of 5 and 6 between 1 and 1000.

4. The eighth term of an arithmetic sequence is 216 and the seventeenth term is 369. Find the:

(a) first term

(b) common difference

(c) sum of the first 40 terms.

5. The first term of an arithmetic series is 28. The common difference is 6.

(a) Find the sum of the first 20 terms of the series

(b) The sum of the first n terms of the series is 5800. Show that n satisfies the equation

$$3n^2 + 25n - 5800 = 0$$

(c) Hence solve the equation to find n.

Solving practical problems by summing arithmetic series

Arithmetic series appear in the real world. An example that you might have come across is simple interest. Simple interest is a type of financial investment used in which an initial monetary value is invested and earns interest. The amount of money increases in a straight line as the rate of increase stays the same. For example, if Luca put \$500 into a saving account that earned 1% interest a year, she would earn \$5 a year ($1\% \times 500 = 5$). So, at the end of the first year she would have \$500 + \$5 = \$505. At the end of the second year she would have \$505 + \$5 = \$510. At the end of 5 years she would have \$500 + \$5 + \$5 + \$5 + \$5 + \$5 = \$525 in her savings account. It can also be useful to know long it will be before a particular total is reached. You can apply the formulae learned in this chapter to find out this kind of information.

Make sure that you start the solution to each problem by writing down the values that you know and can identify what it is that you need to find.

Worked example 3.7

Q. Oscar decides that he will save an extra €2 each week. He saves €6 the first week and €8 the next.

(a) How much will he have saved over the first 13 weeks?

(b) How long will it take him to save €300?

A. (a) $u_1 = 6, d = 2$

> Start by writing down what you know and then what you need to find out: we have the first term ($u_1 = 2$) and the common difference (+2). We need to find the sum of the amount saved in the first 13 weeks (this is denoted by S_{13}).

$$S_{13} = \frac{13}{2}[2\times6+(13-1)\times2]$$
$$= \frac{13}{2}[12+24]$$
$$= 13\times18$$
$$= 234$$

Oscar will save €234 over the first 13 weeks.

> Using algebra: because we don't know the last term (the amount Oscar saved in week 13), we use the formula $S_n = \frac{n}{2}[2u_1 + (n-1)d]$ for the sum of an arithmetic sequence and substitute in the values we have.

$u_1 = 6, d = 2$

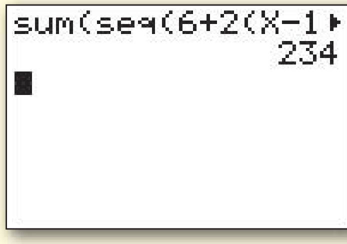
$u_n = 6 + 2(n-1)$

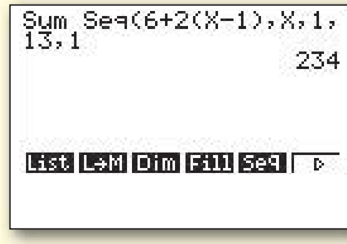
> Check the answer using your GDC.
>
> Substitute the values of u_1 (6) and d (2) into the formula for the nth term of an arithmetic sequence: $u_n = u_1 + (n-1)d$.

continued . . .

Use the 'sum' and 'seq' functions on your GDC. See page 659 of the GDC chapter if you need a reminder.

TEXAS **CASIO**





The answer matches.

Oscar will save €234 over the first 13 weeks.

We know the first term and the common difference. This time we want to find the number of terms in the sequence (n) when the sum of the sequence (S_n) is 300.

(b) $u_1 = 6, d = 2$

Using algebra: we can substitute the values we know into the appropriate formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}[2u_1 + (n-1)d]$.

$$S_n = \frac{n}{2}[2\times6+(n-1)\times2]$$

$$300 = \frac{n}{2}[2\times6+(n-1)\times2]$$

Rearrange the equation to solve for n.

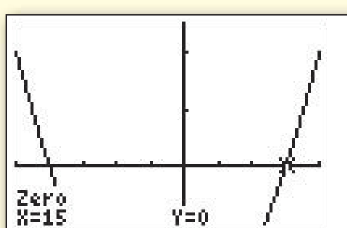
$$300 = \frac{n}{2}[2\times6+(n-1)\times2]$$
$$= \frac{n}{2}[12+2n-2]$$
$$= \frac{n}{2}[10+2n]$$
$$= 5n+n^2$$

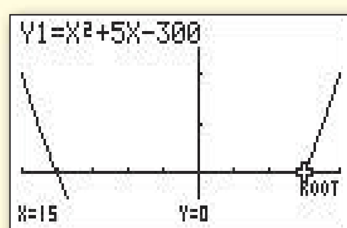
This is a quadratic equation. Solve using your preferred method.

RR See Chapter 2 for methods of solving quadratic equations.

$$n^2+5n-300=0$$

TEXAS **CASIO**





$n = 15$ or $n = -20$

continued . . .

Remember that n is a number of weeks, so it must be a positive integer. Therefore the solution of -20 is not possible.

It will take Oscar 15 weeks to save €300.

Check using algebra.

By factorising:

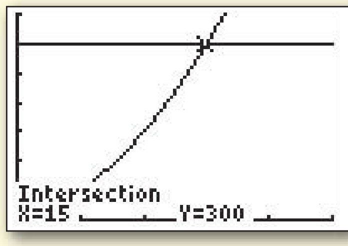
$n^2 + 5n - 300 = 0$

$(n - 15)(n + 20) = 0$

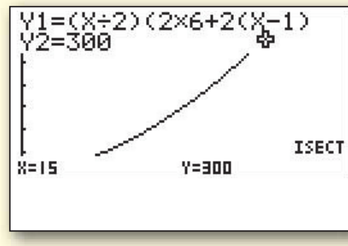
$n = 15$

You could also have used your GDC by plotting the graph of $S_n = \frac{n}{2}[2u_1 + (n-1)d]$ and the line $y = 300$ on the same axis to find the point of intersection. (See '*2.3 (a) Solving quadratic equations using a graph*' on page 656 of the GDC chapter if you need to.)

TEXAS



CASIO



The graphs intersect at $n = 15$. The answers match.

It will take Oscar 15 weeks to save €300.

The German mathematician Carl Friedrich Gauss (1777–1855) was a child prodigy, able to correct his father's arithmetic at the age of three. His teacher at the village school in Braunschweig, Lower Saxony, found it very hard to find sums to occupy him. One day, he asked Gauss to add up all the numbers from 1 to 100. Gauss was only seven years old, but within minutes he had the answer: the first one hundred natural numbers add up to 5050. He had used the same pattern explained in Learning links 3B to add up an arithmetic series:

$$\begin{array}{ccccccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & 4 & + & \dots & + & 98 & + & 99 & + & 100 \\ S & = & 100 & + & 99 & + & 98 & + & 97 & + & \dots & + & 3 & + & 2 & + & 1 \end{array}$$

This gives 100 pairs of numbers that add up to 101. But the series has been used twice, so $S_{100} = \frac{1}{2} \times 100 \times 101 = 5050$. Gauss became one of the most influential and innovative mathematicians of his time, and his ideas still inspire mathematicians today.

Is this a proof of Gauss's method of summing an arithmetic series? Or does a mathematical proof have to use algebra? It is said that mathematical proof and scientific proof are quite different. Try to think of ways in which they are alike, and ways in which they differ.

Exercise 3.8

1. Mrs Gomez has decided to save towards her daughter Alejandra's university education. She will pay €400 into a savings account on Alejandra's sixth birthday, then €550 on her seventh birthday, and so on, increasing the deposit by €150 each year. When Alejandra is sixteen, the last deposit will be made, and all the interest that has accumulated will be added.

(a) How much will Mrs Gomez pay into the account on Alejandra's tenth birthday?

(b) What will be the total amount of money in the account immediately after Alejandra's twelfth birthday?

(c) What is the total amount of money Alejandra can expect to have in her savings account before interest is added?

2. Ali and Husain are recruiting students into a new Mathematics Club at their school. Each week they plan to recruit two more new members than in the previous week. In the first week they gained three new members; the next week they recruited five new members, the week after that seven new members, and so on.

(a) Show that they can expect to recruit 21 new members in the 10th week.

(b) What is the expected **total** membership of the Club in the 20th week (excluding Ali and Husain)?

(c) In which week is the total membership (excluding Ali and Husain) expected to exceed 80?

3. Carmen has decided to save money over a period of two years. She saves \$1 in the first week, \$3 in the second week, \$5 in third week, and so on, with her weekly savings forming an arithmetic series.

(a) Find the amount that she saves in the last week of the first year.

(b) Calculate her total savings over the complete two-year period.

4. Adebayo started receiving pocket money when he was twelve years old. His first monthly pocket money was 2000 nairas. This increased each month by 250 nairas.

(a) Calculate the total pocket money he received in the first year.

Adebayo has been saving all his pocket money so far. After three years of saving he decides to buy himself a treat for 60,000 nairas.

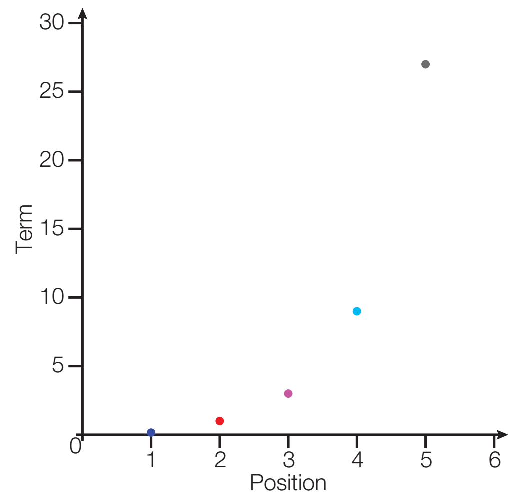
(b) How much of his savings over the three years will remain after he spends the 60,000 nairas?

3.3 Geometric sequences

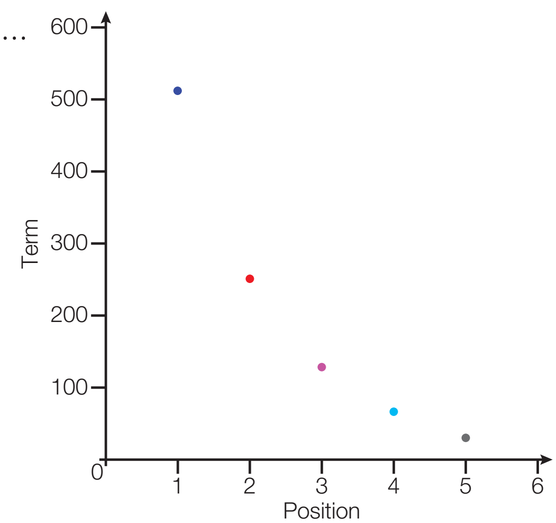
A geometric sequence is a list of numbers with a pattern, but this pattern is different from that of an arithmetic sequence.

Recall these example sequences from the beginning of section 3.1.

3. $\frac{1}{3}$, 1, 3, 9, 27, …



5. 512, 256, 128, 64, 32, …



The graphs of these sequences are curves rather than straight lines. The curve is increasing in the case of sequence 3 and decreasing in the case of sequence 5.

Such sequences are called **geometric sequences**: the number pattern starts at a particular value and is then **multiplied**, or **divided**, by the same amount each time. This fixed multiplier from each term to the next is called the **common ratio** of the geometric sequence.

Look closer at sequence 3:

The sequence starts at $\frac{1}{3}$, and is multiplied by 3 for each subsequent term.

$$\frac{1}{3} \xrightarrow{\times 3} 1 \xrightarrow{\times 3} 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27$$

Dividing each term by the value of the term before it, we get
$\frac{1}{1/3} = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = 3$, so **3** is the common ratio of this sequence.

Now look at sequence 5:

The sequence starts at 512, and is multiplied by $\frac{1}{2}$ for each subsequent term.

$$512 \xrightarrow{\times\frac{1}{2}} 256 \xrightarrow{\times\frac{1}{2}} 128 \xrightarrow{\times\frac{1}{2}} 64 \xrightarrow{\times\frac{1}{2}} 32$$

$\frac{256}{512} = \frac{128}{256} = \frac{64}{128} = \frac{32}{64} = \frac{1}{2}$, so $\frac{1}{2}$ is the common ratio of this sequence.

Exercise 3.9

1. Which of the following sequences are geometric sequences?

(a) 2, 22, 222, 2222, …

(b) 1, 10, 100, 1000, …

(c) 1, 4, 16, 25, 36, …

(d) $84, 56, 37\frac{1}{3}, 24\frac{8}{9}, \ldots$

(e) $5, 5^2, 5^3, 5^4, \ldots$

(f) $3 + y, 6 + y, 12 + y, 24 + y, \ldots$

(g) $2, -4a, 8a^2, -16a^3, \ldots$

(h) $x^3, x^6, x^9, x^{12}, \ldots$

2. Find the next three terms in each of the following geometric sequences.

(a) 4, 8, 16, __, __, __, …

(b) 200, 120, 72, __, __, __, …

(c) 729, 243, 81, __, __, __, …

(d) 30.5, 36.6, 43.92, __, __, __, …

3. Find the missing terms in the following geometric sequences.

(a) 480, 360, __, 202.5, __, …

(b) __, __, 12.5, 25, 50, …

(c) −6, __, __, −162, −486, …

(d) $1, \frac{1}{5}$, __, __, __, …

The *n*th term of a geometric sequence

The pattern in a geometric sequence can lead to a formula for calculating the value of any term. The formula for the general term of a geometric sequence, known as the *n*th term, is given by:

$u_n = u_1 \times r^{n-1}$, where r is the common ratio.

For example, the sequence 512, 256, 128, 64, 32, … has:

- $u_1 = 512$
- $u_2 = 512 \times (\frac{1}{2})^1 = 256$
- $u_3 = 512 \times (\frac{1}{2})^2 = 128$
- $u_4 = 512 \times (\frac{1}{2})^3 = 64$
- and $u_7 = 512 \times (\frac{1}{2})^{7-1} = 512 \times (\frac{1}{2})^6 = 8$

If you know the first term and the common ratio, you can use the formula for the general term to calculate the value of any term in the sequence.

Worked example 3.8

Q. Consider the sequence 2, 3, 4.5, …

(a) Confirm that this is a geometric sequence.

(b) Write down the values of u_1 and r.

(c) Calculate u_7.

Check the ratio between consecutive terms; is it constant?

A. (a) $\frac{3}{2} = 1.5$, $\frac{4.5}{3} = 1.5$

The ratio is the same, so this is a geometric sequence.

u_1 is the first term and r is the common ratio between terms.

(b) $u_1 = 2$ and $r = 1.5$

Substitute known values into the formula: multiply the first term by the common ratio (7 – 1 =) 6 times.

(c) $u_7 = 2 \times (1.5)^6 = 22.8$ (3 s.f.)

You can use the formula to find other values too, depending on what information you start with. For instance, if you know the value of two terms of a geometric sequence, you can use the formula to find the common ratio and the first term.

Worked example 3.9

Q. In a geometric sequence, the second term is 2 and the fifth term is −16. Find:

(a) the common ratio

(b) the first term.

There are two ways you can approach this problem.

A. Method 1:

Draw the sequence.

(a) ? 2 ? ? −16

$u_2 = 2$ and u_5 is −16.

To get from u_2 to u_5 you need to multiply by the common ratio 3 times. We can write this as an equation and solve for r.

$$2 \times r^3 = -16$$
$$r^3 = -8$$
$$r = -2$$

To go from the 1st term to the 2nd term, you multiply by r; so to go from the 2nd term to the 1st term, divide by r.

(b) $u_2 = 2, r = -2$

$$2 \div -2 = -1$$
$$u_1 = -1$$

Method 2:

Use the general formula $u_n = u_1 \times r^{n-1}$. For $u_2 = 2$ substitute $n = 2$ into the formula. For $u_5 = -16$ substitute $n = 5$ into the formula to give two equations.

(a) $u_n = u_1 \times r^{n-1}$

$$u_2 = u_1 \times r = 2 \ (1)$$
$$u_5 = u_1 \times r^4 = -16 \ (2)$$

RR ***Return to Chapter 2 for methods of solving pairs of linear equations if you need to.***

We have a pair of linear equations. Here, we solve by elimination. Divide equation (2) by equation (1) to find r.

$$u_1 r = 2$$
$$u_1 r^4 = -16$$
$$\frac{u_1 r^4}{u_1 r} = \frac{-16}{2}$$
$$r^3 = -8$$
$$r = -2$$

Substitute the value of r into either one of the equations above.

(b) $u_1 \times (-2) = 2$

$$u_1 = 2 \div (-2) = -1$$
$$u_1 = -1$$

Learning links

3D Deriving the formula for the *n*th term of a geometric sequence

Take the sequence $\frac{1}{3}$, 1, 3, 9, 27, 81, …

The first term is $\frac{1}{3}$, and the common ratio is 3.

Write out the terms in a way that helps you to spot a pattern:

		Or	What is done to u_1
The first term	$u_1 = \frac{1}{3}$	$u_1 = \frac{1}{3}$	
The second term	$u_2 = \frac{1}{3} \times 3 = 1$	$u_2 = \frac{1}{3} \times 3$	Multiplying by one 3
The third term	$u_3 = 3 \times 1 = 3$	$u_3 = \frac{1}{3} \times 3 \times 3$	Multiplying by two 3s
The fourth term	$u_4 = 3 \times 3 = 9$	$u_4 = \frac{1}{3} \times 3 \times 3 \times 3$	Multiplying by three 3s

The pattern shows that if you want to find the value of the *n*th term in the sequence, u_n, take the value of the first term, u_1, and multiply by the common ratio, r, $(n - 1)$ times, that is, one time fewer than the term's position.

Exercise 3.10

1. Consider the sequence 10, 15, 22.5, …

(a) Confirm that this is a geometric sequence.

(b) Write down the values of u_1 and r.

(c) Calculate u_{10} to 1 decimal place.

2. (a) In a geometric sequence, $u_3 = 3$ and $u_7 = 48$. Find r, u_1 and u_{10}.

(b) In a geometric sequence, $u_3 = 3$ and $u_6 = 81$. Find r, u_1 and u_{10}.

3. The second term of a geometric sequence is 5 and the fourth term is 20.

(a) Find the first term and the common ratio.

(b) Use these values to calculate the 12th term.

4. For each of the following geometric sequences, find the common ratio and the specified term.

   (a) 3, 12, 48, … ; 10th term

   (b) 64, 96, 144, … ; 20th term

   (c) 90, 288, 921.6, … ; 18th term

   (d) 1, 1.5, 2.25, … ; 16th term

   (e) 180, −198, 217.8, … ; 12th term

   (f) −45, −99, −217.8, … ; 21st term

5. The following sequences are all geometric. For each, calculate the common ratio and three more terms.

   (a) 2, 3, … (b) −1, 2, …

   (c) 100, 50, … (d) 1, 1.1, …

Finding the position of a term in a geometric sequence

You can determine the position of a term by any of the following methods:

- drawing a diagram
- trial and improvement
- using your GDC to plot graphs
- using your GDC's equation solver.

This is the same as asking: what is the position of the term 1458 in the sequence?

Worked example 3.10

Q. How many terms are there in the geometric sequence 2, 6, 18, … , 1458?

Drawing a diagram: you know that $u_1 = 2$ and $r = 3$, so you can write successive terms until you reach 1458.

A. Method 1:

$u_1 = 2$ and $r = 3$

2, 6, 18, 54, 162, 486, 1458

so there are 7 terms.

continued . . .

Method 2:

$u_1 = 2$ and $r = 3$

$u_n = u_1 \times r^{n-1}$

$u_n = 2 \times 3^{n-1} = 1458$

Try

$n = 5$: $2 \times 3^4 = 162$, too low

$n = 8$: $2 \times 3^7 = 4374$, too high

$n = 7$: $2 \times 3^6 = 1458$, correct answer

So 1458 is the 7th term.

Use **trial and improvement** by substituting different values of n into the formula.

exam tip

Make sure you write down all your working.

Method 3:

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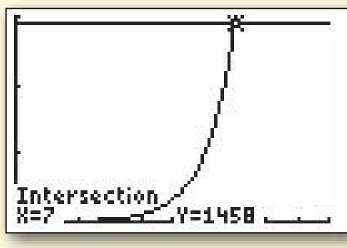

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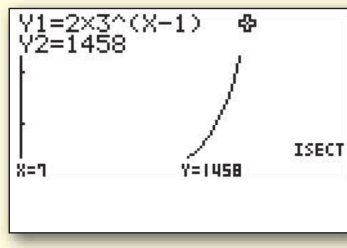


From GDC: $n = 7$

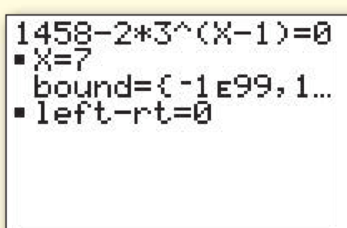
Use your GDC to plot graphs: you want to find the value of n when $2 \times 3^{n-1} = 1458$. Plot the graphs of $y = 2 \times 3^{n-1}$ and $y = 1458$ and find the point that the two graphs intersect. See *'19.2 (a) Solving unfamiliar equations using a graph'* on page 684 of the GDC chapter for a reminder of how to do this if you need to.

If you got your answer from your GDC, say so when you write down your answer.

Method 4:

$2 \times 3^{n-1} = 1458$

TEXAS



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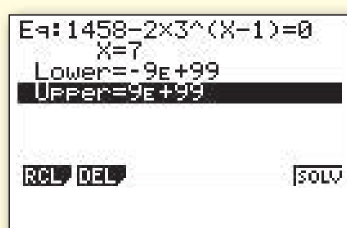


From GDC: $n = 7$

Use your GDC's equation solver: enter the equation into your GDC and solve. See section *'19.2 (b) Solving unfamiliar equations using an equation solver'* on page 685 of the GDC chapter if you need a reminder of how to do this.

Exercise 3.11

1. Use the equation solver on your GDC to find the number of terms in each geometric sequence.

   (a) 2, 4, 8, … , 512

   (b) 1.2, 1.44, … , 2.0736

   (c) 1600, 160, 16, … , 0.016

   (d) 81, 27, 9, … , $\frac{1}{27}$

   (e) 1, 0.1, 0.01, … , 0.00001

   (f) $1, \frac{3}{4}, \frac{9}{16}, \ldots, \frac{729}{4096}$

2. Use the graphing function on your GDC to find the number of terms in each of the following geometric sequences.

	First term	Common ratio	Last term
(a)	100	0.8	40.96
(b)	24	1.5	273.375
(c)	8	2	8192
(d)	40	$\frac{1}{2}$	$\frac{5}{128}$
(e)	160	$\frac{1}{4}$	$\frac{5}{524288}$
(f)	243	$\frac{1}{3}$	$\frac{1}{243}$

3. The first term of a geometric sequence is 80 400 and the common ratio is 1.05. Given that $u_n = 97\,726.7025$, use the method of trial and improvement to find the value of n.

4. For each of the following geometric sequences you are given the first term, the common ratio and u_n. Use an appropriate method to determine the value of n.

   (a) $u_1 = 8, r = 2, u_n = 16\,384$

   (b) $u_1 = 3, r = 6, u_n = 839\,908$

   (c) $u_1 = 4, r = \frac{1}{2}, u_n = 0.0625$

   (d) $u_1 = 256, r = \frac{1}{4}, u_n = \frac{1}{64}$

   (e) $u_1 = \frac{8}{81}, r = \frac{3}{2}, u_n = \frac{27}{16}$

   (f) $u_1 = -48, r = 2, u_n = -1536$

3.4 Geometric series: the sum of a geometric sequence

If you add up the terms of a geometric sequence you get a **geometric series**.

Geometric series are important in financial calculations as well as in many other practical situations.

You can calculate the sum of a geometric series by adding all the values together but as you are multiplying by a common ratio each time, the sum can get very large very quickly and it is much more convenient to use a formula.

There are two formulae for the sum of a geometric series

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

exam tip

The two formulae give exactly the same answer, but it is easier to use $S_n = \frac{u_1(r^n - 1)}{r - 1}$ if $r > 1$ and $S_n = \frac{u_1(1 - r^n)}{1 - r}$ if $r < 1$.

hint

The ≠ symbol in the formula above means 'not equal to'. Here r cannot equal 1 because this would make the denominator zero, and you cannot divide by zero.

There is a legend about a Chinese emperor who wished to reward the wise man who had invented the game of chess. The emperor promised the wise man whatever he desired. The wise man asked for 'one grain of rice on the first square of the chess board, two grains on the second and four grains on the third, until the chessboard is filled'. How much rice is that in total? Was it a sensible promise for the emperor to have made?

Learning links

3E Deriving the formula for the sum of a geometric sequence

Suppose that a university student, who did not have very much money, asked his grandmother to help him out with a very small amount of money each day. He thought she wouldn't miss a few cents each day but that if he saved it, it would help him a lot.

continued . . .

He asked her to give him:

1 cent on 1 April

2 cents on 2 April

4 cents on 3 April …

and to continue this pattern until the end of the month (30 days).

The amounts of money given each day form a geometric sequence with $u_1 = 1$ and $r = 2$. Looking at the first few terms it looks like his idea is a very good one, it won't cost his grandmother that much money at all.

How much would his grandmother have given him by the end of April?

The sum after 30 days can be calculated by adding up the amounts he got each day.

$$S = 1 + 2 + 4 + 8 + 16 + \ldots$$

$$S = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{29}$$

To add up a geometric series, we can use the pattern in the numbers. If the value of each term in the series is doubled and then written below the original series, you can see that most of the terms are the same.

$$\begin{aligned} S &= 1 + 2 + 4 + 8 + 16 + \ldots + 2^{28} + 2^{29} \quad (1) \\ 2S &= \phantom{1 + {}} 2 + 4 + 8 + 16 + 32 + \ldots + 2^{29} + 2^{30} \quad (2) \end{aligned}$$

So, subtracting one sequence from the other makes most terms cancel out and gives you a simple way of calculating the answer. (2) – (1) gives:

$$2S - S = (\cancel{2} + \cancel{4} + \cancel{8} + \cancel{16} + \ldots + \cancel{2^{29}} + 2^{30}) - (1 + \cancel{2} + \cancel{4} + \cancel{8} + \cancel{16} + \ldots + \cancel{2^{29}})$$

That is, $S = 2^{30} - 1$ (where 2^{30} is r^n and 1 is u_1).

So $S = 1073741823$ cents, which is 10,737,418.23 dollars!.

Luckily for her, the student's grandmother was a mathematician and she recognised the geometrical pattern and disagreed with his idea!

We can use a similar trick to calculate the sum of a general geometric series: multiply the series by its common ratio and write the result below the original series; then subtract the two series, which makes most terms cancel and leads to the following two formulae for the sum: $S_n = \dfrac{u_1(r^n - 1)}{r - 1} = \dfrac{u_1(1 - r^n)}{1 - r}$, $r \neq 1$

Worked example 3.11

Q. Consider the geometric sequence −1, 2, −4, 8, −16, …

(a) Write down the values of u_1 and r.

(b) Find the tenth term.

(c) Find the sum of the first ten terms.

A. (a) $u_1 = -1,\ r = -2$

Use the formula $u_n = u_1 \times r^{n-1}$. Substitute in the values from (a), and substitute in $n = 10$.

(b) $u_n = u_1 \times r^{n-1}$

$u_n = (-1) \times (-2)^{n-1}$

$u_{10} = (-1) \times (-2)^9 = (-1) \times (-512) = 512$

(c)

Use the formula $S_n = \dfrac{u_1(r^n - 1)}{r-1}$. Substitute in the values of u_1, r, and $n = 10$.

$$S_n = \frac{(-1)((-2)^n - 1)}{(-2)-1}$$

$$S_{10} = \frac{(-1)((-2)^{10} - 1)}{(-2)-1} = 341$$

You can use either of the formulae $\dfrac{u_1(r^n - 1)}{r-1}$ or $\dfrac{u_1(1-r^n)}{1-r}$, but if you use the second version, be especially careful when entering the negative signs into your calculator.

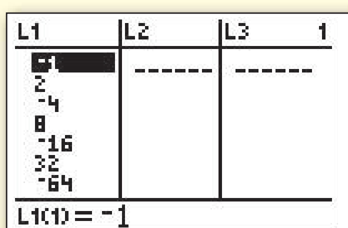
$$S_n = \frac{(-1)(1-(-2)^n)}{1-(-2)}$$

$$S_{10} = \frac{(-1)(1-(-2)^{10})}{1-(-2)}$$

$$= \frac{(-2)^{10} - 1}{3} = 341$$

Check using your GDC. See section '*3.3 Finding the sum of the geometric series using the list function*' on page 661 of the GDC chapter.

TEXAS



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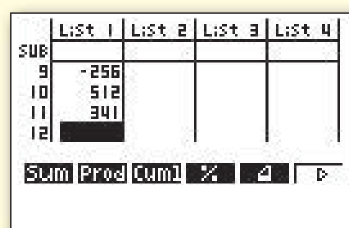

From GDC: Sum = 341

Worked example 3.12

Q. Consider the geometric sequence 3, 2.4, 1.92, 1.536, …

(a) Write down the value of u_1.

(b) Show that $r = 0.8$.

(c) Calculate S_8.

A. (a) $u_1 = 3$

(b) $r = 2.4 \div 3 = 0.8$

Calculate the ratio using the second and first terms (you could have calculated the ratio between any two consecutive terms).

(c) $S_n = \dfrac{u_1(1-r^n)}{1-r} = \dfrac{3(1-0.8^n)}{1-0.8}$

$S_8 = \dfrac{3(1-0.8^8)}{1-0.8} = 12.5$ (3 s.f.)

$r < 1$, so use the formula $S_n = \dfrac{u_1(1-r^n)}{1-r}$

(Alternatively, you could use the list function on your GDC, as per the check in Worked example 3.11.)

Exercise 3.12

1. Use the formula $S_n = \dfrac{u_1(r^n - 1)}{r-1}$ to calculate the sum of each of the following geometric series.

(a) $u_1 = 2, r = 2$; first 10 terms

(b) $u_1 = 0.3, r = 5$; first 8 terms

(c) $u_1 = 1.465, r = 7$; first 11 terms

(d) $u_1 = \dfrac{5}{8}, r = 4$; first 9 terms

(e) $u_1 = 1, r = 1.2$; first 12 terms

(f) $u_1 = 3.6, r = 2.06$; first 21 terms

2. Use the formula $S_n = \dfrac{u_1(1-r^n)}{1-r}$ to calculate the sum of the first 20 terms of each of these geometric series.

(a) $u_1 = 500, r = 0.2$

(b) $u_1 = 1200, r = \dfrac{1}{4}$

(c) $u_1 = 4, r = -\dfrac{3}{4}$

(d) $u_1 = 84, r = -\dfrac{1}{2}$

(e) $u_1 = 20.6, r = 0.565$

(f) $u_1 = 800, r = 0.01$

3. For the following geometric series you are given the first term (u_1), the common ratio (r), and the number of terms (n). Calculate the sum of the series in each case.

   (a) $u_1 = 4$, $r = 2$, $n = 10$

   (b) $u_1 = 16$, $r = 1.5$, $n = 18$

   (c) $u_1 = -81$, $r = 0.9$, $n = 20$

   (d) $u_1 = 1.25$, $r = 3$, $n = 22$

   (e) $u_1 = 358$, $r = 0.95$, $n = 30$

4. For each of the following geometric series, find the common ratio and hence calculate the sum of the specified number of terms.

   (a) $10 + 20 + 40 + \dots$; first 20 terms

   (b) $3 + 9 + 27 + \dots$; first 24 terms

   (c) $128 + 64 + 32 + \dots$; first 18 terms

   (d) $4 + 4.8 + 5.76 + \dots$; first 16 terms

   (e) $6.25 + 1.25 + 0.25 + \dots$; first 10 terms

   (f) $2^3 + 2^4 + 2^5 + \dots$; first 20 terms

   (g) $7^3 + 7^6 + 7^9 + \dots$; first 24 terms

5. The third term of a geometric sequence is 144 and the sixth term is 9.216. Given that all the terms in the sequence are positive, calculate:

   (a) the common ratio

   (b) the sum of the first 16 terms of the sequence.

6. Take the geometric series $1000 + 500 + 250 + \dots$

   (a) Write down u_1 and r.

   (b) Calculate the sum of the first 10 terms.

7. The first three terms of a geometric series are $32 + 16 + 8 + \dots$. Calculate the sum of the first 14 terms. Give your answer three decimal places.

8. A geometric series has $u_2 = 15$ and $u_4 = 135$.

   (a) Write down the values of u_1 and r.

   (b) Calculate the sum of the first 16 terms. Write down all the figures on your GDC screen.

   (c) Give the answer to three significant figures.

   (d) Write the answer to part (c) in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

hint

$\in$ is set notation and means 'belongs to', so $k \in \mathbb{Z}$ means that k is within the set of integers denoted by $\mathbb{Z}$.

Solving practical problems involving geometric sequences and series

Many practical problems can be solved using geometric sequences and series, especially in situations where there is growth or decay that does not follow a straight line ('nonlinear').

FF You will see more examples in Chapter 4 and Chapter 19.

Worked example 3.13

Q. The heating is switched off in a school laboratory. An experiment is set up to look at the growth in a population of fruit flies. As the temperature in the laboratory is low, it is found that the population is only growing by 30% per day.

On Monday there are 20 fruit flies.

(a) How many fruit flies will there be on Friday?

(b) How long will it be before the population exceeds 500 fruit flies?

A. $r = 130\% = 1.3$

The population growing by 30% per day means that each day the population is (100 + 30)% times the previous day's population.

(a) $u_n = u_1 \times r^{n-1} = 20 \times 1.3^{n-1}$

If Monday is the first day, then Friday is the fifth day and so $n = 5$.

$u_5 = 20 \times 1.3^4 = 57.122 \approx 57$

Substitute known values into the general formula. Round the answer to the nearest integer (the number of fruit flies is discrete data and so must be a whole number).

(b) $20 \times 1.3^{n-1} > 500$

We are looking for the smallest n for which $u_n > 500$.

As in Worked example 3.10, there are four main ways in which you could do this: we show three of them.

Trial and improvement.

continued . . .

Method 1:

Try

$n = 10$: $20 \times 1.3^9 = 212$ (3 s.f.)

$n = 15$: $20 \times 1.3^{15} = 787$ (3 s.f.)

$n = 14$: $20 \times 1.3^{13} = 606$ (3 s.f.)

$n = 13$: $20 \times 1.3^{12} = 466$ (3 s.f.)

So $n = 14$ is the first day on which the population will exceed 500.

Use your GDC as in Worked example 3.10 to plot $y = 20 \times 1.3^{x-1}$ and $y = 500$. Find the x-value of their intersection and round up to the next integer.

Method 2:

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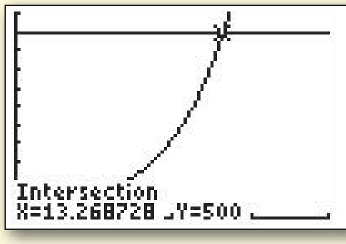

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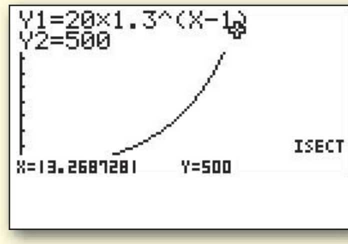


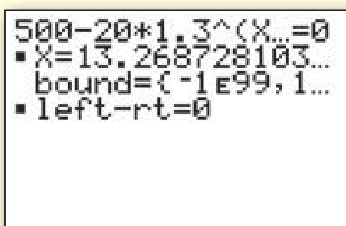
From GDC: intersection at $x = 13.3$

So $n = 14$

Use the equation solver on your GDC as you did in Worked example 3.10 using the equation $20 \times 1.3^{n-1} = 500$.

Method 3:

TEXAS



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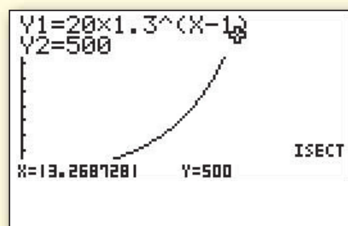


From GDC: solution is $x = 13.3$

So $n = 14$

Exercise 3.13

1. The first time that Sim counts a population of beetles there are 20. A month later there are 35 beetles. The population grows as a geometric sequence. How many beetles are there after 12 months? Give your answer to three significant figures.

2. A business makes a profit of 50,000 AUD in 2005. In 2006, the profit is 60,000 AUD. If the profit continues to grow as a geometric sequence, what is the profit in 2010? Give your answer to three significant figures.

3. The Kumars lived in a rented four-bedroom property from January 2000 until December 2009.

   (a) Given that their yearly rent formed a geometric sequence, complete the table below.

Year	Annual rent (£)
2000	8990
2001	9170
2002	
2003	
2004	
2005	
2006	
2007	
2008	
2009	

   (b) What was the total amount of rent paid over the 10-year period?

4. Miles joined the Apollo Golf and Gym Club in January 2005. The annual fee was £672 then. Since 2005 the fees have increased steadily by 4% every year.

   (a) How much was the membership fee in 2010?

   (b) How much in total did Miles pay in membership fees from 2005 to 2010?

Summary

You should know

- what an arithmetic sequence and series is
- what a geometric sequence and series is
- how to use the formulae for the nth term and the sum of the first n terms, of an arithmetic and a geometric sequence
- some common applications of arithmetic and geometric sequences and series in the real world.

Mixed examination practice

Exam-style questions

1. The first three terms of an arithmetic sequence are 24, 41, 58, …

(a) State the common difference.

(b) Work out the 20th term of the sequence.

(c) Find the sum of the first 20 terms of the sequence.

2. The 5th term of an arithmetic sequence is 42 and the 9th term is 64.

(a) Write two equations involving the first term u_1 and the common difference d.

(b) Solve the equations to find the first term and the common difference.

3. Jasmine has been collecting stamps for a while. She collected 7 stamps in the first month, 11 stamps in the following month, 15 stamps in the month after that, and so on, in an arithmetic sequence.

(a) How many stamps did she collect in the 12th month?

(b) What was her **total** collection after 24 months?

(c) How many more stamps did she collect altogether in her third year than in her second year?

(d) After how many months will her total collection exceed 500 stamps?

4. The first term of a geometric sequence is 400 and the fourth term is 204.8.
All the terms are positive numbers.

(a) Find the common ratio.

(b) Find the sum of the first 18 terms.

5. A ball is dropped onto a hard surface. It bounces up 2 m the first time. Each bounce after that reaches a height that is 85% of the one before. What height will the ball reach on the seventh bounce?

6. Marthe starts a savings account for her son. On his first birthday she puts €120 in the account, on his second birthday she deposits €126, and on his third birthday €132.30.

(a) Explain why the common ratio for this geometric series is 1.05.

(b) How much money will Marthe put in her son's account on his fifth birthday? On his tenth birthday?

(c) How much money will he have in his account at the end of ten years?

Note: The answers to (b) and (c) are financial, so remember to round them to two decimal places.

7. Tomasz is training for a 50 km bicycle race. He cycles 5 hours in the first week, and plans to increase the training time by 10% each week.

   (a) Show that he will cycle for 5.5 hours the second week.

   (b) How long will he cycle in the fifth week?

   (c) He trains for 12 weeks. What is his total training time?

   Give all your answers to three significant figures.

8. Mei is starting a new job. Her starting salary is $30,000, and she is told that it will increase each year. She can choose one of the following:

   Option 1: an annual increase of $500

   Option 2: an increase of 2% each year

   Mei plans to stay in the job for five years. Which option should she choose?

9. (i) Francine is repaying a loan she took out to buy a car. Her monthly repayments form an arithmetic series. She repaid $450 in the first month, $445 in the second month, $440 in the third month, and so on.

   (a) How much will Francine repay in the 30th month?

   (b) What is the total amount she will have repaid after three years?

   (ii) Bradley bought an identical car at the same time as Francine. However, his loan repayments were different. He repaid $600 in the first month, $592 in the second month, $584 in the third month, and so on.

   (a) What is the total amount Bradley will have repaid after three years?

   (b) In which month will the repayment amount be the same for Francine and Bradley?

10. Super Bricks is a new building company. The company produced 140 000 bricks in the first month. The volume of production is expected to rise at a monthly rate of 8%.

   (a) What is the expected monthly volume of production at the end of the first year of production?

   (b) What is the estimated total volume of production over the first twelve months?

   A rival company Brick Works produced 250 000 bricks in the first month. Responding to a rise in demand, the company plans to increase production at a monthly rate of 4%.

   (c) Show that over the next six months, Brick Works will produce more than 1.6 million bricks.

Past paper questions

1. The first three terms of an arithmetic sequence are

$2k+3$, $5k-2$ and $10k-15$.

(a) Show that $k=4$. *[3 marks]*

(b) Find the values of the first three terms of the sequence. *[1 mark]*

(c) Write down the value of the common difference. *[1 mark]*

(d) Calculate the 20th term of the sequence. *[2 marks]*

(e) Find the sum of the first 15 terms of the sequence. *[2 marks]*

[Total 9 marks]

[Nov 2006, Paper 2, Question 4(i)] (*© IB Organization 2006*)

2. A geometric progression G_1 has 1 as its first term and 3 as its common ratio.

(a) The sum of the first n terms of G_1 is 29 524. Find n. *[3 marks]*

A second geometric progression G_2 has the form $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

(b) State the common ratio for G_2. *[1 mark]*

(c) Calculate the sum of the first 10 terms of G_2. *[2 marks]*

(d) Explain why the sum of the first 1000 terms of G_2 will give the same answer as the sum of the first 10 terms, when corrected to three significant figures. *[1 mark]*

(e) Using your results from parts (a) to (c), or otherwise, calculate the sum of the first 10 terms of the sequence $2, 3\frac{1}{3}, 9\frac{1}{9}, 27\frac{1}{27}, \ldots$

Give your answer **correct to one decimal place**. *[3 marks]*

[Total 10 marks]

[May 2007, Paper 2, Question 4(ii)] (*© IB Organization 2007*)

3. The first term of an arithmetic sequence is 0 and the common difference is 12.

(a) Find the value of the 96th term of the sequence. *[2 marks]*

The first term of a geometric sequence is 6. The 6th term of the geometric sequence is equal to the 17th term of the arithmetic sequence given above.

(b) Write down an equation using this information. *[2 marks]*

(c) Calculate the common ratio of the geometric sequence. *[2 marks]*

[Total 6 marks]

[May 2008, Paper 1, Question 8] (*© IB Organization 2008*)

4. A National Lottery is offering prizes in a new competition. The winner may choose one of the following.

Option one: \$1000 each week for 10 weeks.

Option two: \$250 in the first week, \$450 in the second week, \$650 in the third week, increasing by \$200 each week for a total of 10 weeks.

Option three: \$10 in the first week, \$20 in the second week, \$40 in the third week, continuing to double for a total of 10 weeks.

(a) Calculate the amount you receive in the tenth week, if you select:

(i) option two;

(ii) option three. *[6 marks]*

(b) What is the total amount you receive if you select **option two**? *[2 marks]*

(c) Which option has the greatest total value? Justify your answer by showing all appropriate calculations. *[4 marks]*

[Total 12 marks]

[**May 2002, Paper 2, Question 2**] (*© IB Organization 2002*)

Chapter 4 Financial mathematics

In this chapter you will learn:

- about currency conversions
- about simple and compound interest
- how to use geometric sequences and series in a financial context
- about annual inflation and depreciation.

The stock market.

The investment of money, the trading of money, and the best way of looking after your money — these are important concerns all over the world, and understanding them helps people to make sensible decisions about their lives.

This chapter demonstrates how your knowledge about numbers and sequences can help you gain a better understanding of money matters.

4.1 Currency conversions

A currency is the system of money in use in a particular country. Over time, different countries have developed different currencies with distinctive names and values that depend on the history and geography of that country.

When you travel from one country to another, you will need to change (or convert) one currency into another.

The value of a country's currency relative to other currencies can also affect the trade and prosperity of that country.

- In some countries, the basic unit of currency is divided into smaller units. For example, £1 = 100 pence, \$1 = 100 cents. So answers to financial questions involving such currencies should be given to **two decimal places**, for example as \$5.34.
- In other countries, the basic unit of currency is not split into smaller parts; examples include the Japanese yen. So answers to financial questions involving such currencies should be given to the **nearest whole number**.

Some common currencies are listed in the table below, along with their symbols.

Currency	Three-letter abbreviation	Symbol
Australian dollar	AUD	\$
Canadian dollar	CAD	\$
European euro	EUR	€
Hong Kong dollar	HKD	\$
Indian rupee	INR	₹
Japanese yen	JPY	¥
US dollar	USD	\$
UK pound	GBP	£

Currency conversions are very important in economics — for example, in the trading of commodities.

Currency exchange rates

To change one currency to another you need to know the **exchange rate**. The exchange rate is also called the 'foreign-exchange rate', 'forex rate' or 'FX rate'. It is the ratio between the values of two currency units, a number that you can use to exchange one currency for another.

exam tip

In financial questions set in examinations, you will be expected to give answers to two decimal places, unless you are given different instructions. If you forget to do this, you will be given a one-mark penalty for that examination paper.

Exchange rates vary constantly; they can be affected by trade, the politics and stability of a country, and natural catastrophes.

Newspapers and websites provide up-to-date tables of rates. An example of an exchange-rate table, taken from http://www.x-rates.com/, is shown on page 112.

Notice that, in the table on page 112, the exchange rates are given to six significant figures. You can use these numbers to calculate the amount that you will pay or receive in a transaction, but for most currencies you would express the final amount to two decimal places.

This particular table has the 'from' currencies arranged in columns and the 'to' currencies arranged in rows. This means that the number in the first column and second row of the table, 0.775915, is the exchange rate from CAD to EUR, that is, the number of Canadian dollars for each Euro. Be aware that other tables may be arranged differently, with the 'from' currencies in rows and the 'to' currencies in columns.

What effects might fluctuations in a country's currency value have on international trade? For example, if a currency becomes relatively expensive, it could make that country's exports more difficult to sell.

To work out the amount of currency that you will get when you exchange currency 1 for currency 2, you multiply by the exchange rate:

amount of currency 2 = amount of currency 1 × exchange rate

	CAD	EUR
CAD	1	1.2888
EUR	0.775915	1

Wednesday, January 12, 2011

Modified from http://www.x-rates.com

Worked example 4.1

Q. (a) Zara is planning to travel from France to Canada. She has €1800 to change into Canadian dollars. How many dollars can she buy if she exchanges the currencies according to the table alongside?

(b) During her holiday, Zara spends $1600 of her Canadian dollars. If she changes her remaining dollars back into euros when she gets home, according to the table above, how many euros will she have?

In the table above, the exchange rate from EUR to CAD (4th column, 3rd row) is 1 EUR = 1.2888 CAD

A. (a) 1800 EUR = 1800 × 1.2888 CAD
= 2319.84 CAD (2 d.p.)

First, calculate how much money she has left.

(b) \$2319.84 − \$1600 = \$719.84

The exchange rate from CAD to EUR (3rd column, 4th row) is 1 CAD = 0.775915 EUR

719.84 CAD = 719.84 × 0.775915 EUR
= 558.53465 EUR

So she will get €558.53 (2 d.p.)

Exercise 4.1

1. Use the given exchange-rate table to answer the following questions. Round your answers to two decimal places.

	USD	GBP	CAD	EUR	AUD
USD	1	1.57137	1.01378	1.30656	0.993413
GBP	0.636387	1	0.645161	0.831483	0.632195
CAD	0.9864	1.55	1	1.2888	0.979903
EUR	0.765363	1.20266	0.775915	1	0.760322
AUD	1.00662	1.58178	1.0205	1.31523	1

Wednesday, January 12, 2011

***Source*: http://www.x-rates.com**

(a) How many CAD can be exchanged for 1200 AUD?

(b) How many GBP can be exchanged for 1500 CAD?

(c) How many AUD can be exchanged for 800 USD?

(d) How many EUR can be exchanged for 750 GBP?

(e) How many USD can be exchanged for 2450 EUR?

2. Yuki is travelling from Japan to Canada on business. In Tokyo the bank is quoting an exchange rate of 1 JPY = 0.01205 CAD.

(a) If she changes ¥120,000 into Canadian dollars, how many dollars will she receive?

(b) She spends $885 in Canada. How many dollars does she have left?

(c) She changes her Canadian money back into Japanese yen when she returns. Assuming that the exchange rate has remained the same, how many yen does she have now? (Give your answer to the nearest yen.)

3. Jean is going on holiday. He is travelling from Geneva to Sicily. He takes 950 CHF (Swiss francs) with him, and changes it into euros at his hotel.

The hotel exchange rate is 1 CHF = 0.76715 EUR.

(a) How many euros does he receive?

(b) At the end of the two-week holiday he has €125 left. How many Swiss francs will he get back? Two weeks later the hotel exchange rate is 1 CHF = 0.821567 EUR.

Buying and selling currency

Currency can be bought and sold like any other commodity. If you visit a bureau de change, you will see the current rates of exchange displayed on a board like this.

Country	Currency	We buy	We sell
Australia	AUD	1.697	1.460
Euro zone	EUR	1.299	1.132
Hong Kong	HKD	12.89	11.34
Japan	JPY	139.7	121.0
Switzerland	CHF	1.638	1.411
USA	USD	1.683	1.472

The rates in this table were taken from a British newspaper published in 2011, so they are relative to GBP (Great Britain Pounds). In other words, they give the number of units of each currency for each pound sterling. The 'We **buy**' column gives the rates used when you change other currencies **to** GBP; the 'We **sell**' column gives the rates that apply when you change money **from** GBP to other currencies.

As money itself can be traded like a commodity, there could be global effects if certain currencies are over- or under-valued. Does this have ethical implications?

You can also buy and sell currency through banks and brokers. These are businesses being run for profit, and there are two ways in which they can make a profit from currency exchange:

1. The broker charges **commission**, which is a percentage of the value of the transaction. You pay the commission to the broker for the money that you exchange.
2. The bank buys currency from customers at one rate, and sells it at a lower rate. The difference in price will be the bank's profit.

For example, Marc and Annie are travelling from Cardiff in Wales to Hong Kong. They decide to take £1250 each, but go to different banks to change the currency.

Marc's bank quotes an exchange rate of 1 GBP = 12.39 HKD, and charges 2% commission in the original currency.

This means that the amount of commission he pays is $0.02 \times £1250 = £25$.

The exchange rate is applied to the remaining $£1250 - £25 = £1225$, giving $1225 \times 12.39 = 15{,}177.75$ HKD.

Annie decides to buy from a bank that charges no commission.

The bank is selling at a rate of 1 GBP = 11.46 HKD, so her £1250 becomes $1250 \times 11.46 = 14{,}325$ HKD.

After the trip they return to Wales with 3700 HKD each, and sell their currency back to their respective banks.

Marc's bank again charges 2% commission and uses the same exchange rate as before: 1 GBP = 12.39 HKD.

So the amount of commission he pays is $0.02 \times \$3700 = \$74$, and the amount of money remaining to be exchanged is $\$3700 - \$74 = \$3626$.

Applying the exchange rate then gives $3626 \div 12.39 = 292.66$ GBP.

Annie's bank is buying at a rate of 1 GBP = 13.29 HKD, so her \$3700 becomes $3700 \div 13.29 = 278.40$ GBP.

In this example, even though Marc was charged commission, he had the better deal in both directions of conversion. This will not always be the case, so it is usually worth checking to see where you can get the best deal for a particular transaction.

Exercise 4.2

1. Jung lives in Singapore and is travelling to Thailand. She wants to change 1600 Singapore dollars (SGD) into Thai baht (THB).

   The bank in Singapore charges 2% commission in dollars and quotes an exchange rate of 1 SGD = 23.910 THB.

   (a) Calculate the number of Thai baht that Jung will receive.

   (b) In Bangkok she buys a hat for 85 THB. What did it cost in SGD, according to the exchange rate given above?

2. Cara lives in Scotland and travels to the Netherlands. Her bank quotes her an exchange rate of £1 = €1.17 and charges her £3 in commission. Cara changes £150 into euros.

   (a) How many GBP does Cara have left to exchange after she has paid the commission?

   (b) How many euros does Cara receive from the bank?

   (c) In the Netherlands, Cara buys 750 g of cheese for her grandfather. If the cheese costs €12.80 per kilogram, calculate the cost of this gift in GBP.

For questions 3–5, use the exchange rates in the following table.

Question 3		Question 4		Question5	
USA/EUR exchange		EUR/GBP exchange		GBP/HKD exchange	
We buy	We sell	We buy	We sell	We buy	We sell
0.75863	0.72596	0.87504	0.83736	12.3768	11.8438

3. Mike travels from the USA to Spain. He changes $900 into euros at his bank.

   (a) Use the table above to calculate the number of euros that he receives, assuming that no commission is charged.

   Mike's flight is cancelled, and he changes the euros back into dollars.

   (b) Using the same table, how many dollars does the bank give him back?

   (c) How much money has Mike lost by changing his money twice?

4. Anya travels from Greece to England. She needs to change €1200 into GBP. She has a choice to make.

   (a) Bank A charges 1.6% commission in euros and offers an exchange rate of 1€ = £0.851483. Calculate the number of GBP she would receive from bank A.

   (b) Bank B buys and sells GBP at the rates shown in the table above, with no commission charged. Calculate the number of GBP that Anya would receive from bank B.

   (c) Which bank should Anya choose?

5. Nic is flying from England to Hong Kong.

   (a) He changes £240 into Hong Kong dollars. If the bank is selling at the rate shown in the table above, how many HKD does he receive?

   (b) On his return to England, he has 780 HKD left to change back into pounds. If the bank is buying at the rate shown in the table above, how many GBP does he receive?

4.2 Compound interest

"That piggy bank is for my college fund."

exam tip

You need to study simple interest in order to understand compound interest, but the examiners will not be setting questions on simple interest.

Investment, and the earning of interest from investments, happens all over the world. However, it is important to realise that not all societies have the same attitude to financial investments and treat them in the same way. For example, Islamic law does not allow the charging of interest or fees for loans of money. Therefore Islamic banking operates in a different way.

If you have saved or earned some money, you will want to keep it in a safe place. It would be even better if, while that money is being looked after, it generates more money and your original sum increases.

The sum of money that you deposit (pay into the bank) is called the **capital**. You can earn **interest** on that capital while it is in the bank.

There are two types of interest:

- **simple interest**, where the interest is calculated on only the original sum deposited
- **compound interest**, where the interest is calculated on the original sum plus all the interest previously accumulated.

In this course we will focus on compound interest.

Imagine that your friend Shane has saved 800 AUD, and needs to decide on the best way to protect it and use it to finance his future travel plans.

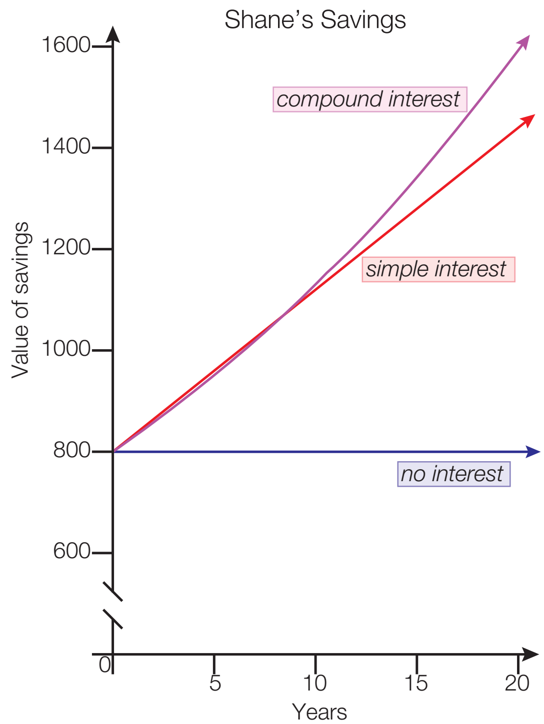
Shane has identified three different options, and asks you for your advice.

- *Option* 1: He puts the money in a safe place at home, and adds any more money that he saves to his original sum.
- *Option* 2: He puts the money in a bank for a year. The bank is offering him an annual **interest rate** of 4% on his deposit. At the end of the year, he can withdraw the interest to spend or leave the deposit in the bank to earn more interest; however, Shane will only ever get interest on the initial deposit. This is an example of simple interest.
- *Option* 3: He puts the money in the bank for several years. The bank offers him a lower interest rate of 3.5% annually, but after one year it calculates the interest, adds the amount to Shane's deposit, and then the following year pays interest on both the original deposit and the interest that it has already earned. This is an example of compound interest.

Let's look at what will happen to Shane's $800 deposit over a few years, assuming that he makes no withdrawals or further deposits (numbers are rounded to the nearest dollar):

Shane's savings	Option 1	Option 2	Option 3
End of year 1	800 AUD	$800 \times 4\% = 32$ $800 + 32 = 832$ AUD	$800 \times 3.5\% = 28$ $800 + 28 = 828$ AUD
End of year 2	800 AUD	$800 + 32 + 32 = 864$ AUD	$828 + (828 \times 3.5\%) = 828 + 29 = 857$ AUD
End of year 3	800 AUD	$800 + 32 + 32 + 32 = 896$ AUD	$857 + (857 \times 3.5\%) = 857 + 30 = 887$ AUD

Putting these numbers on a graph makes it easier to see what will happen in the future.



Shane's dilemma illustrates three different ways of looking after money:

- Option 1 earns no interest on the capital of 800 AUD. The value of the savings does not increase.
- Option 2, **simple interest**, earns interest on the capital, but that interest is not 'reinvested'. The value of the savings increases in a straight line; the rate of increase stays the same.
- Option 3, **compound interest**, earns interest on the capital, and that interest is reinvested by being added to the capital to form a new base for future interest calculations. The value of the savings increases **exponentially**.

FF *You will meet exponential growth in Chapter 19.*

What is your advice to Shane?

Now we look at the mathematical structure of Options 2 and 3 in more detail.

With **simple interest**, as long as you keep the original deposit in the bank, it will continue to earn the same amount of interest year after year.

In Shane's case, his 800 AUD deposit will earn $800 \times 4\% = 32$ AUD of interest every year. So at the end of 5 years he will have earned $5 \times 32 = 160$ AUD interest, and his total balance will be $800 + 32 + 32 + 32 + 32 + 32 = 800 + (5 \times 32) = 960$ AUD.

Note that his total balance from one year to the next can be viewed as an **arithmetic sequence** with first term 800 and common difference 32.

RR *You met sequences in Chapter 3.*

In general, using algebra, if an amount of money PV (which stands for **present value**) is invested for n years at an annual rate of interest r%, then the amount of simple interest earned, is $\frac{PV \times r \times n}{100}$, and the total balance is given by $PV + \frac{PV \times r \times n}{100} = PV\left(1+\frac{rn}{100}\right)$.

Applying these formulae to Shane's case, we take $PV = 800$, $r = 4$ and $n = 5$. So:

$$\text{interest earned} = \frac{800 \times 4 \times 5}{100} = 160 \text{ AUD}$$

$$\text{total balance} = 800 \times \left(1+\frac{4 \times 5}{100}\right) = 960 \text{ AUD}$$

With **compound interest**, as long as you make no withdrawals, the amount of interest earned will increase from year to year.

This is easier to understand if we do the interest calculations year by year. In Shane's case:

	Total balance
Beginning of year 1	800 AUD
End of year 1	$800 + 800 \times \frac{3.5}{100} = 828$ AUD
End of year 2	$\left(800 + 800 \times \frac{3.5}{100}\right) + \left(800 + 800 \times \frac{3.5}{100}\right) \times \frac{3.5}{100} = 856.98$ AUD

In general, using algebra, we can work out a formula for the account balance after n years.

If an amount of money PV is invested for n years at an annual rate of interest r%, then:

	Total balance	Formula
Beginning of year 1	PV	PV
End of year 1	$PV + PV \times \frac{r}{100} = PV\left(1+\frac{r}{100}\right)$	$PV\left(1+\frac{r}{100}\right)$
End of year 2	$PV\left(1+\frac{r}{100}\right) + PV\left(1+\frac{r}{100}\right) \times \frac{r}{100}$ $= PV\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right)$ (by taking out a factor of $PV\left(1+\frac{r}{100}\right)$ from the first line)	$PV\left(1+\frac{r}{100}\right)^2$
End of year n		$PV\left(1+\frac{r}{100}\right)^n$

Note that the total balance from one year to the next can be viewed as an **geometric sequence** with first term PV and common ratio $\left(1+\frac{r}{100}\right)$.

The total balance after n years is referred to as the **future value** of the investment, denoted by FV.

The general formula for compound interest is:

The formulae for simple and compound interest are important in finance and economics, for example in forecasting economic growth.

$$FV = PV\left(1+\frac{r}{100k}\right)^{kn}$$

where FV = future value, PV = present value, n = number of years, $r\%$ = annual rate of interest, and k = number of compounding periods per year.

So far we have considered only the situation where interest is calculated at the end of each year, that is, $k = 1$. But often interest is calculated at more frequent intervals, such as quarterly ($k = 4$) or monthly ($k = 12$).

Applying the formula to Shane's example, we take $PV = 800$, $r = 3.5$, $k = 1$, and $n = 5$. Then we get:

$$FV = 800\times\left(1+\frac{3.5}{100}\right)^{5} = 950.15 \text{ AUD}$$

The interest that Shane will have earned over five years is $950.15 - 800 = 150.15$ AUD.

Here are some important points to remember when calculating compound interest:

- Think about when you are investing the money; this is usually taken to be the beginning of year 1.
- Think about when you are being asked for the total balance or interest earned; this is usually at the end of a certain year, which is the 'n' in the formula.
- Look at the time intervals at which interest is calculated; interest may be calculated yearly, half-yearly, quarterly, monthly or daily.

Note that if Shane invests his \$800 at the same compound interest rate but calculated quarterly or monthly rather than yearly, he will end up with different amounts at the end of five years.

Think carefully about compound interest. Would it be good for your savings or pension, but bad for your credit card? Do you always read the small print?

Investing 800 AUD at 3.5% compounded quarterly yields:

$$FV = 800\times\left(1+\frac{3.5}{100\times 4}\right)^{4\times 5} = 800\times\left(1+\frac{3.5}{400}\right)^{20} = 952.27 \text{ AUD}$$

Investing 800 AUD at 3.5% compounded monthly yields:

$$FV = 800\times\left(1+\frac{3.5}{100\times 12}\right)^{12\times 5} = 800\times\left(1+\frac{3.5}{1200}\right)^{60} = 952.75 \text{ AUD}$$

By compounding more frequently, a little more interest has been earned.

The compound interest formula $FV = PV\left(1+\frac{r}{100k}\right)^{kn}$ can be used to solve many of the financial questions that you are likely to meet.

Worked example 4.2

Q. Twins Paco and Peta are given 12,000 pesos each by an aunt.

(a) Paco decides to invest his money in a bank that is offering 4.8% interest compounded yearly. How much money will he have in the bank after five years?

(b) Peta decides to invest her money in another bank that is offering 4.6% interest compounded every three months (quarterly). How much money will she have in the bank after five years?

(c) Which twin has made the better investment?

Use the compound interest formula with $PV = 12000$, $r = 4.8$, $k = 1$ and $n = 5$.

A. (a) Paco: $FV = 12000 \times \left(1+\frac{4.8}{100}\right)^5 = 15170$ pesos

Use the compound interest formula with $PV = 12000$, $r = 4.6$, $k = 4$ and $n = 5$.

(b) Peta: $FV = 12000 \times \left(1+\frac{4.6}{100\times 4}\right)^{4\times 5}$

$= 12000 \times \left(1+\frac{4.6}{400}\right)^{20} = 15083$ pesos

(c) Paco has made the better investment.

Worked example 4.3

Q. Jan invests 950 euros in a bank that pays 3.8% interest compounded yearly. How many years will it take for his investment to double?

Write down the formula for compound interest and substitute in the given values. Here, $k = 1$.

A. $FV = PV\left(1+\frac{r}{100k}\right)^{kn}$

$FV = 950\left(1+\frac{3.8}{100}\right)^n$

We want to find the value of n for which $FV = 2 \times PV$.

$2\times 950 = 950\left(1+\frac{3.8}{100}\right)^n$

After dividing both sides of the equation by 950, we have an equation to solve for n.

continued . . .

$2 = (1 + 0.038)^n$

exam tip

Logarithms are not in the syllabus, but if you are confident in using them, you can use them to solve equations like this one.

There are several ways to find n. Here are two.

Method 1:

Trial and improvement.

Try

$n = 20$: $(1 + 0.038)^{20} = 2.108$, too high

$n = 18$: $(1 + 0.038)^{18} = 1.957$, too low

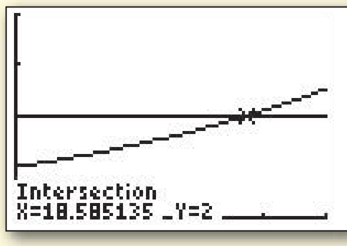
$n = 19$: $(1 + 0.038)^{19} = 2.031$, close to 2

It will take 19 years for Jan's investment to double.

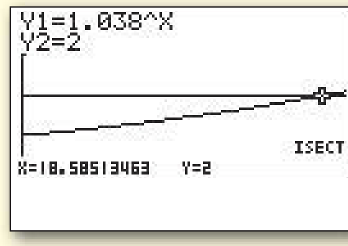
Use your GDC to plot the graphs of $y = (1 + 0.038)^x$ and $y = 2$ and look for their intersection. See section '*19.2(a) Solving unfamiliar equations using a graph*' on page 684 of the GDC chapter if you need reminding how.

Method 2:

TEXAS



CASIO



Intersection is at $x = 18.6$

So it will take 19 years.

Exercise 4.3

1. Astrid has inherited 40,000 euros from her great aunt. She has decided to invest the money in a savings account at an interest rate of 6% **per annum**.

   Find how much the investment will be worth after five years if interest is compounded:

   (a) yearly

   (b) quarterly (every three months)

   (c) monthly.

2. Kyle has just retired from his job. He was given a **lump sum** pension of 500,000 AUD and decided to save his money in a deposit account which pays 4% interest rate per annum.

   Find how much his savings will be worth after one year if the interest is compounded:

   (a) quarterly (b) monthly (c) weekly.

3. Richard invests \$3800 in a bank account at r% interest compounded annually. He hopes to save \$5000 in five years. What is the minimum value of r needed for him to meet his target?

4. Mr Woodward saw the following advertisement in a timber investment brochure.

> HIGH RETURN TIMBER INVESTMENTS
> A typical investment:
> £18,000 invested over 12 years projects
> returns of £104,109

He decided to follow up on the opportunity, but wants to know what the annual interest rate is. Calculate the interest rate for him, assuming that interest is compounded yearly.

5. Mrs Simpson invests a lump sum of \$100,000 in an offshore business. She is promised an interest rate of 4.2% compounded annually.

(a) How long will it take for the investment to double?

(b) If it takes n years for the investment to treble, find the value of n.

6. Dr Chapman saw the following advertisement in a newspaper.

> Fix your savings for the next 10 years:
> - Earn a competitive fixed rate of 3.6% per annum
> - Invest from £1000 to £2 million
> - Withdrawals and closures are not allowed
> - Strictly limited offer

He decided to invest £64,000 for the advertised term of 10 years.

(a) Assuming that interest is compounded annually, what will his investment be worth after 5 years?

(b) Will he be able to double his investment over the 10-year period? Justify your answer.

4.3 The GDC and financial mathematics

exam tip

This is one of the programs that you are allowed to use in IB examinations. If it is not on your GDC already, you should be able to download it from the website for your brand of GDC.

Most GDCs have a built-in program that can help you to solve the financial problems that you might come across in exercises, examinations and real life. See '*4.1 The financial App, TVM*' on page 661 of the GDC chapter for how to use this App.

Once you have loaded the program, you will see a screen containing a list of variables. You need to read the question carefully and input the values for the different variables.

The letters and symbols used in the program should be familiar to you if you have already solved financial problems using the compound interest formula:

N or n = number of time periods (usually years)

I% = interest rate

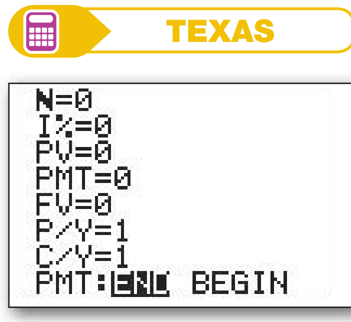
PV = present value

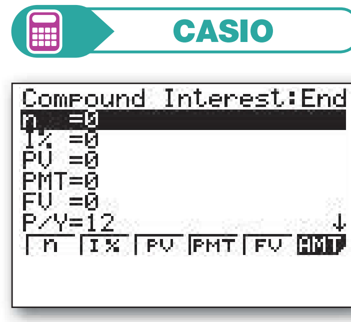
PMT = extra payments into the account each year

FV = future value

P/Y = number of interest payments made into the account each year

C/Y = number of compounding periods each year





Keep the following points in mind when you use the TVM program:

- Write down the answers to the calculations clearly.
- You must not use 'calculator language' in projects and examinations, so quote the compound interest formula in your solutions.
- You may find it helpful to list the values required by the program as you read the question.
- For investments, PV must be entered as a **negative** quantity: you have invested the money, so it is no longer in your account — the bank has it!
- To get the answer using a TEXAS GDC, highlight the quantity that you want to find, then press ALPHA, SOLVE.
- To get the answer using a CASIO GDC, press the Function key for the quantity that you want.

hint

You might find it interesting to look up 'compound interest' on the internet. A search will turn up many websites, some explaining the mathematics, and some giving simple examples similar to the ones in this book.

Worked example 4.4

Q. Marie has $680 to invest. Suppose that she puts it in a bank account that pays 3.7% interest compounded annually.

(a) How much money will she have at the end of 10 years?

(b) How long will it take to double her investment?

(c) Marie would like her investment to double in 15 years. At what rate does her account need to earn interest if she is to achieve this?

continued . . .

Substitute the given values into the compound interest formula. Here $k = 1$.

A. (a) $FV = PV\left(1+\frac{r}{100k}\right)^{kn}$

$FV = 680\left(1+\frac{3.7}{100}\right)^{n} = 680\times 1.037^{n}$

Substitute in $n = 10$.

$FV = 680\times(1+0.037)^{10} = 977.90$

Check the answer using the TVM program on your GDC.

TEXAS

CASIO

We want to find the value of n for which $FV = 2\times PV$, so use the compound interest formula to write down the equation that we need to solve.

(b) $2\times 680 = 680\times(1+0.037)^{n}$

Divide both sides by 680.

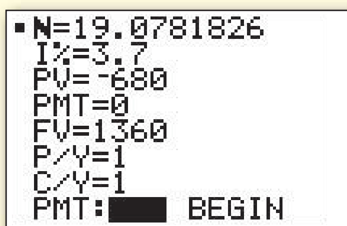
$2 = (1+0.037)^{n}$

Solve the equation by trial and improvement or by plotting graphs and finding their intersection.

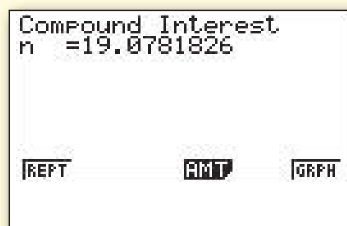
From GDC: $n = 19.08$

Check the answer using the TVM program on your GDC.

TEXAS



CASIO



Now r is the unknown value that we want to find.

(c) $FV = 680\left(1+\frac{r}{100}\right)^{n}$

We want FV to be $2\times PV$ when $n = 15$, so write down the equation to solve for r.

$2\times 680 = 680\left(1+\frac{r}{100}\right)^{15}$

Divide both sides by 680.

$2 = \left(1+\frac{r}{100}\right)^{15}$

continued . . .

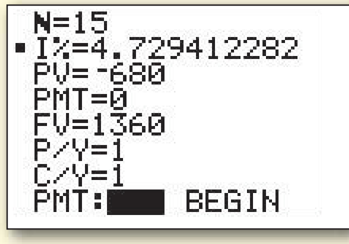
Solve the equation by trial and improvement or plotting graphs and finding their intersection.

From GDC: $r = 4.73$

So the account needs to earn interest at 4.73%.

Check your answer using the TVM program.

TEXAS



CASIO

As you can see, the TVM program is particularly useful when solving questions like (b) and (c) in Worked example 4.4.

Worked example 4.5

Q. In part (a) of Worked example 4.4, Marie invested her money at 3.7% compound annually. She could also invest her money at a rate of 3.6% compounded monthly.

What would be the difference in her investment amounts under the two schemes after 10 years?

Substitute the given values into the compound interest formula.

A. $FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$

$FV = 680\left(1 + \frac{3.6}{100 \times 12}\right)^{12n} = 680 \times 1.003^{12n}$

Substitute in $n = 10$.

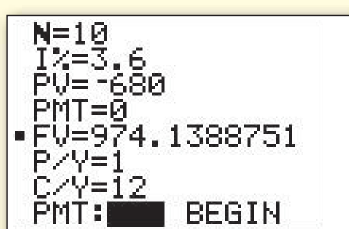
$FV = 680 \times (1 + 0.003)^{120} = 974.14$

Calculate the difference between this answer and the answer to Worked example 4.4(a).

$£977.90 - £974.14 = £3.76$

Check your answer using the TVM program.

TEXAS



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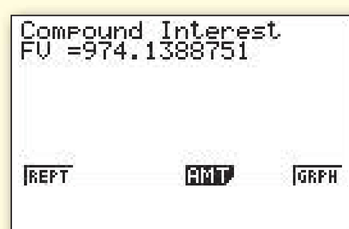


$977.90 - 974.14 = 3.76$ GBP

hint

There are also many websites that contain 'compound interest calculators', which can help you to predict how investments will change over time depending on different interest rates or different compounding periods. They can also work out the regular payment amounts needed to repay a loan.

The GDC's TVM program is also very useful for solving loan-related problems, such as working out the monthly repayment for a mortgage or a car-purchase loan. In this case, the program's variables have the following meaning:

- N or n = number of time periods (instalments)
- I% = interest rate
- PV = total loan amount, which should be entered as a **positive** number
- PMT = payment amount per period; this is the quantity you want to find
- FV = **0**, because at the end of the lifetime of the loan, you want to owe nothing
- P/Y = number of payments per year
- C/Y = number of interest compounding periods per year, usually the same as P/Y

For example, suppose Lukas needs to take out a loan for €9000, and the local bank offers him a loan at 4.5% interest per annum, to be paid back over 5 years in monthly instalments.

Entering $n = 5 \times 12$, I% = 4.5, PV = 9000, FV = 0, P/Y = 12 and C/Y = 12 into his GDC, Lukas finds that PMT = 167.79. This means that his monthly repayment amounts will be €167.79.

The total amount he will have repaid at the end of the five years (60 instalments) is €167.79 × 60 = €10,067.23. So the total interest that the bank will earn from him is €1067.23.

Note that if Lukas had waited until the end of the five years to pay everything back, interest would have accrued on all of the €9000 for five years. Assuming the interest is compounded annually, he would have owed $9000\left(1+\frac{4.5}{100}\right)^5 = 9000 \times 1.045^5$ = €11,215.64 at the end of five years, to be paid off in one go (of which €2215.64 is interest). By repaying in monthly instalments, each month Lukas reduces the outstanding debt on which interest accrues, so overall he ends up paying less; this process of gradually decreasing a debt by paying it back in regular instalments is called **amortisation**.

Exercise 4.4

Try using your GDC to answer the investment-related questions in Exercise 4.3.

The following questions are all concerned with loan repayment problems.

The table on page 127 shows monthly repayment amounts for a loan of £5000 (APR = annual percentage rate). Use the information from the table to answer questions 1–3.

	8.4% APR	9.9% APR	12.9% APR
36 months	£156	£160	£166
48 months	£122	£125	£132
60 months	£101	£104	£111
90 months	£74	£77	£85
120 months	£60	£64	£72
180 months	£48	£52	£60
240 months	£42	£46	£55
300 months	£38	£43	£53

1. (a) Find the monthly repayment on a loan of £5000 at 9.9% per annum taken over 120 months.

(b) Calculate the total amount to be repaid on a loan of £35,000 taken over 60 months at a rate of 12.9% per annum. (*Hint*: £35,000 is seven times £5000.)

2. Two friends, Arthur and Ken, both took out loans of £40,000. Arthur was offered his loan at 9.9% per annum, to be repaid over 60 months.

(a) Work out Arthur's monthly repayment.

(b) Calculate Arthur's total repayment on his loan over the 60 months.

Ken's loan was at 8.4% per annum over 90 months.

(c) Work out Ken's monthly repayment.

(d) Which of the two friends had the better deal? Explain your answer.

3. Margaret is buying a car for £25,000. She pays a 10% deposit and takes out a loan at 8.4% per annum over 48 months.

(a) How much deposit did she pay?

(b) How much did she borrow?

(c) Work out her monthly repayment.

(d) Calculate the total interest paid on the loan.

4. Mr and Mrs Alonso are planning to go on a cruise to celebrate their silver wedding anniversary. They need an extra €4380 to complete their payment, and decide to ask their bank for a loan. They agree to repay over 5 years at 6.8% per annum.

(a) Calculate the value of the monthly repayment.

(b) Find the total amount repaid over the five years.

(c) Find the total amount of interest charged.

(d) Would they have got a better deal if, for the same loan amount of €4380, they had opted for repayment over 3 years at 7.2% per annum? Justify your answer.

5. Jeremy is looking for a new car and wants to buy either a Nissan Pathfinder or a Nissan X-Trail. He printed the following advertisements from a website. However, some of the data was not legible.

	MONTHLY PAYMENTS	CUSTOMER DEPOSIT	CASH PRICE	TOTAL AMOUNT OF CREDIT	TOTAL AMOUNT PAYABLE	DURATION
	?	?	£29,580	£13,958	£31,066	36 months

	MONTHLY PAYMENTS	CUSTOMER DEPOSIT	CASH PRICE	TOTAL AMOUNT OF CREDIT	TOTAL AMOUNT PAYABLE	DURATION
	£539	£17,024	£34,560	?	?	36 months

(a) For the Nissan X-Trail, calculate the monthly payments and the customer deposit.

(b) For the Nissan Pathfinder, calculate the total amount Jeremy needs to borrow and the total amount payable on the loan.

The following table shows monthly repayments for a \$1000 loan taken over different periods and at different rates. Use the information from the table to answer questions 6 and 7.

Loan term (years)	Monthly repayments per \$1000							
	Annual interest rate							
	6.00%	6.25%	6.50%	6.75%	7.00%	7.25%	7.50%	8.00%
3	30.42	30.54	30.65	30.76	30.88	30.99	31.11	31.34
5	19.33	19.45	19.57	19.68	19.80	19.92	20.04	20.28
10	11.10	11.23	11.35	11.48	11.61	11.74	11.87	12.13
12	9.76	9.89	10.02	10.15	10.28	10.42	10.55	10.82
15	8.44	8.57	8.71	8.85	8.99	9.13	9.27	9.56
20	7.16	7.31	7.46	7.60	7.75	7.90	8.06	8.36
25	6.44	6.60	6.75	6.91	7.07	7.23	7.39	7.72
30	6.00	6.16	6.32	6.49	6.65	6.82	6.99	7.34

6. The Johnsons take out a loan for \$8000 over 5 years at a rate of 7.5% per annum. Calculate:

(a) the value of the monthly repayment

(b) the total amount paid over the five years

(c) the amount of interest charged.

7. Mr and Mrs Freeman are planning to refurbish their house at a cost of \$39,500. They decide to take out two separate loans.

Mr Freeman is offered a loan for \$20,500 at 6.5% per annum over 12 years.

Mrs Freeman is offered a loan for the remaining amount at 7.25% over 10 years.

(a) How much in total will the couple pay for the loan?

(b) What will be the total interest charged on the loan?

(c) Would Mr and Mrs Freeman get a better deal if they went for a third offer of a joint loan for the whole amount of $39,500 at 8% per annum over 10 years? Justify your answer.

8. Yuko takes out a loan for ¥43,000 over three years at a rate of 2.17% per annum. Find:

(a) the value of the monthly repayment

(b) the total amount paid over the three years

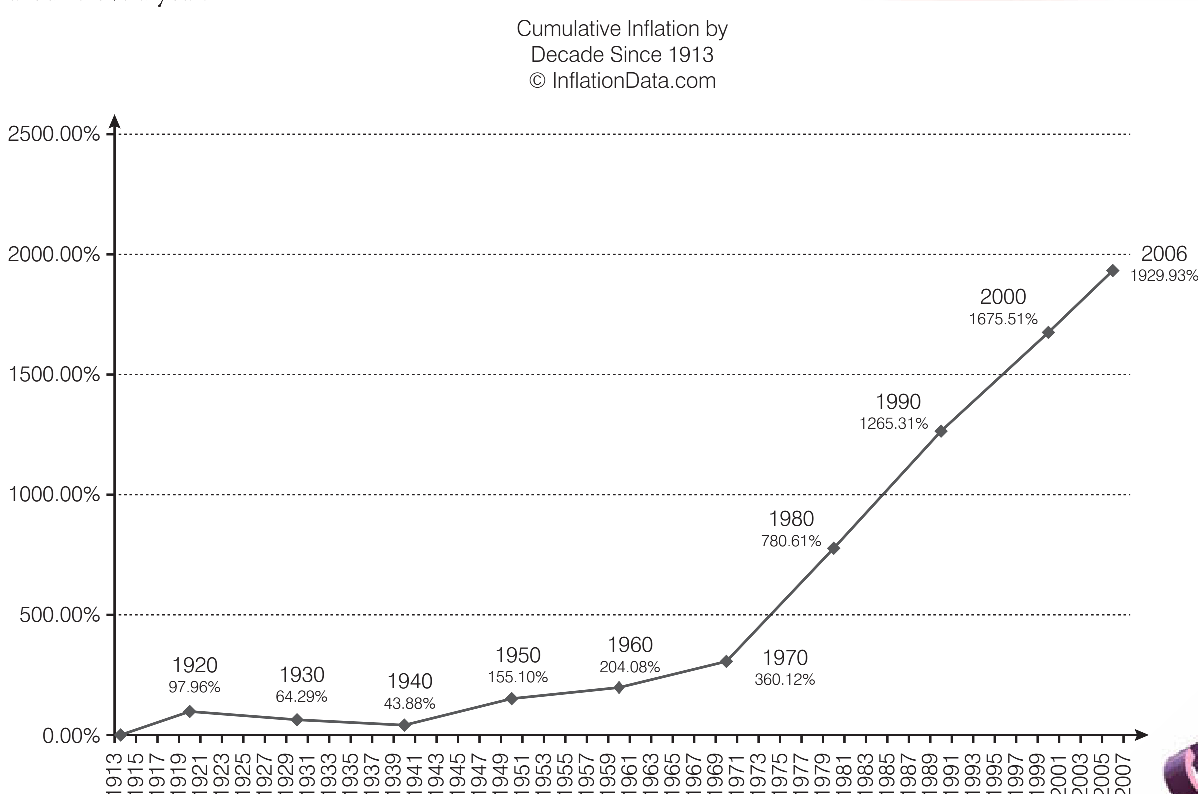
(c) the amount of interest charged.

4.4 Inflation

In England, a litre of milk cost about 19 pence in 1968; it cost approximately 79 pence in 2009.

In America, a bottle of ketchup cost 22 cents in 1966; fifteen years later it cost 99 cents.

The steady rise in prices, and consequent fall in the quantity of goods that the same amount of money can buy, is called **inflation**. Over the past 120 years, inflation in most industrialised countries has been at around 3% a year.



Governments calculate inflation using different items, including the cost of accommodation. Why would they do this?

Inflation rates vary with the price fluctuations of different commodities. Some commodities, such as oil, have a more variable rate than others.

There have been periods of time in certain countries when changes in prices were very drastic. For example, Germany at the time of the Weimar Republic in the 1920s experienced hyperinflation, where prices rose so rapidly that money lost most of its value before it could be spent. In America, following the Wall Street crash of 1929, negative inflation (**deflation**) meant that stock prices dropped suddenly, causing massive unemployment and severe economic depression.

Governments calculate inflation using a statistic called the Consumer Price Index (CPI). This number is estimated by looking at a 'basket' of goods that people use most commonly. The rate at which the cost of those goods in the basket changes over time gives a measure of inflation.

Inflation calculations can be done using the compound interest formula with $k = 1$, $FV = PV\left(1+\frac{r}{100}\right)^n$, or a GDC's TVM program.

Worked example 4.6

Q. In 1990, the average price of a barrel of oil was \$23.19. If price inflation occurred at an average rate of 3% after 1990, what would a barrel of oil have cost in 2010?

Use the compound interest formula. In this case r = annual rate of inflation, and remember that because it is inflation $k = 1$.

A. $FV = PV\left(1+\frac{3}{100}\right)^n$

$FV = 23.19(1+0.03)^{20} = \$41.88$

In 2010, the average price of a barrel of oil was actually \$53.48, even more than expected from a 3% inflation rate.

Worked example 4.7

Q. If the inflation rate in Canada this year is 2.35%, calculate the likely cost of a 450 CAD laptop computer:

(a) one year ago (b) in four years' time.

Substitute the given values into the compound interest formula and set $k = 1$.

A. (a) $FV = PV\left(1+\frac{r}{100}\right)^n$

$= PV\left(1+\frac{2.35}{100}\right)^n = PV \times 1.0235^n$

continued . . .

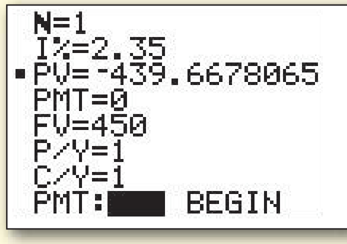
Let PV = price one year ago and FV = price this year. Take n = 1.

$450 = PV \times (1.0235)^1$

$PV = 450 \div 1.0235 = 439.67$ CAD

Check your answer using the TVM program.

TEXAS



CASIO

Substitute known values into the compound interest formula.

This time let PV = price this year, and let FV = price four years later.

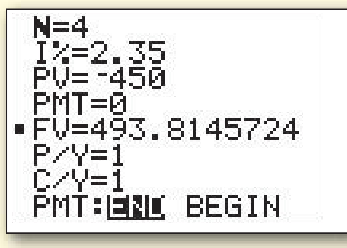
Take n = 4.

(b) $FV = 450 \times (1.0235)^4 = 493.81$ CAD

Check your answer using the TVM program.

Note that the GDC is still treating PV as an investment (hence the negative value), even though you are using the program for a different financial purpose.

TEXAS



CASIO

Exercise 4.5

1. A house cost £198,000 twenty years ago. If the rate of inflation has remained at 4% per year over the past twenty years, find the present cost of the house.

2. Florence bought her car six years ago for €45,850. If inflation has caused the price of cars to increase by 2.8% each year, what would it cost her to buy the car now?

3. Phoenix Communications bought a new IT system for \$3.4 million twelve years ago. If the rate of inflation has been a steady 1.98% per year over the past twelve years, calculate how much the system would have cost them now.

4. The table below shows the average price of selected products in the UK in 1990 and 2004 (information obtained from the Office for National Statistics).

Commodity	Price in £ per kg		Overall rate of inflation, %
	1990	2004	1990–2004
Rump steak, British	8.13	8.97	10.3
Cod fillets	5.74	8.64	?
Sugar, granulated	0.62	0.74	?
Cheese, Cheddar	3.30	5.67	71.8
Apples, eating	1.03	1.25	?
Carrots	0.59	0.57	?

(a) Complete the table by calculating the missing data.

(b) For each of the commodities, work out the annual percentage rate of inflation, assuming it stayed constant over the 1990–2004 period.

5. In 1980 the price of 100 g of instant coffee was \$1.01. As a result of inflation, the price of coffee increased by 3.16% per year. Find how long it took for the price of 100 g of instant coffee to rise to \$1.88.

6. The average price of a litre of petrol in 1990 was 44 cents. After n years, the price increased to 81 cents. Assuming that the steady year-on-year inflation rate was 6.29%, find n.

4.5 Depreciation

If you buy a new bicycle, does it gain value with time? Or does it lose value? Most manufactured goods lose value as they get older. So a bicycle that cost 20,000 INR (Indian rupees) could be worth only 16,500 INR three years later. This reduction in value is called **depreciation**.

Depreciation is important to individuals because they would generally have to sell an item for much less than they paid for it originally. It is important to businesses, because any new equipment that they buy will be worth less from year to year.

As with inflation, you can calculate depreciation using the compound interest formula with $k = 1$, which is $FV = PV\left(1+\frac{r}{100}\right)^n$. But note that as the value is decreasing each year, the interest rate r will be **negative**.

Worked example 4.8

Q. Reena buys a new bicycle for 25,000 INR. She knows that the value of the bicycle will depreciate by 6% each year. What will her bicycle be worth in eight years' time?

Substitute the given values into the formula, remembering that r is negative and $k = 1$.

A. $$FV = PV\left(1+\frac{r}{100}\right)^n$$

$$= PV\left(1+\frac{-6}{100}\right)^n = PV \times 0.94^n$$

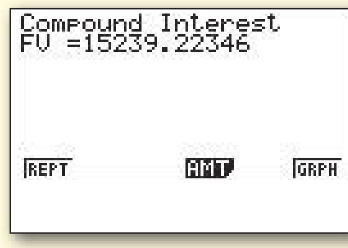
Substitute in $PV = 25000$ and $n = 8$.

$$25000 \times 0.94^n = 15239 \text{ INR}$$

Check your answer using the TVM program.

TEXAS

CASIO



Worked example 4.9

Q. José runs a printing business in Chile. He decides to buy a new printing press at a cost of 4500 pesos. The value of the press depreciates at a rate of 10% each year. How long will it take before the press is worth half the amount that José paid for it?

Substitute the given values into the formula.

Again, r is negative, as the value of the printing press is decreasing and $k = 1$.

A. $$FV = PV\left(1+\frac{r}{100}\right)^n$$

$$= 4500\left(1+\frac{-10}{100}\right)^n = 4500 \times 0.9^n$$

We want to find the value of n for which $FV = 0.5 \times PV$.

$$0.5 \times 4500 = 4500 \times 0.9^n$$

continued . . .

Divide the equation on both sides by $PV = 4500$.

$0.5 = 0.9^n$

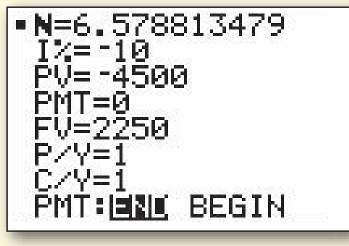
Solve the equation by trial and improvement or plotting graphs on your GDC.

From GDC: $n = 6.58$

So it will take 7 years for the value to halve.

Check your answer using the TVM program.

TEXAS



CASIO

Exercise 4.6

1. A laser printer was bought new for £454.80. The printer depreciates in value at 18% per year.

   (a) Find the printer's value after three years.

   (b) How much of its value will be lost after five years?

2. Christopher bought his car in May 2005 for \$49,995. The rate of depreciation is estimated to be a steady 21% per year.

   (a) How much was Christopher's car worth in May 2010?

   (b) How much of the car's original value did he lose?

   (c) Is it worth selling his car in May 2012 for \$10,000? Justify your answer.

3. I buy a car for €43,000 and keep it for n years. The value of the car after n years is €22,016. If the car's value has been depreciating at a constant rate of 21% per annum, find the value of n.

4. A drinks company bought new laboratory equipment for \$168,000. It is estimated that at the end of its useful life, the value of the equipment will be \$22,750. If the yearly rate of depreciation is 17.5%, find the length of the useful life of the equipment.

5. A company buys a new communication system for \$350,000. It is estimated that for the first three years, the rate of depreciation will be 20% per annum. After three years the rate of depreciation changes to a constant r% per annum, until the system is scrapped for \$12,000 at the end of ten years' service.

   (a) Work out the estimated value of the system after three years.

   (b) Find the value of r.

   (c) How long did it take for the system to lose half of its value?

Summary

You should know:

- how to carry out currency conversions
- what compound interest is and that it can be calculated yearly, half-yearly, quarterly or monthly
- that compound interest is an application of geometric sequences and series in a financial context
- how to use the GDC's financial package (TVM) to answer finance-related questions
- the concepts of annual inflation and depreciation.

Mixed examination practice

Exam-style questions

1. Frederique is travelling from Canada to Moscow. She changes 800 Canadian dollars (CAD) into Russian rubles (RUB). The exchange rate is 1 CAD = 29.7044 RUB.

   (a) How many rubles does she receive?

   On her return Frederique has 7000 rubles left. She decides to change them back into Canadian dollars.

   (b) Assuming that the exchange has remained the same, how many CAD will she receive for her remaining rubles?

2. Andriano invested \$9100 in a five-year investment scheme at the beginning of 2001. Interest was compounded monthly at an annual rate of 7%.

   (a) What was the investment worth after two years?

   (b) How much overall interest did Andriano receive at the end of the five-year period?

3. A school minibus cost £23,500 when bought new in August 2008. Its value depreciated at an annual rate of 18% over the next three years.

   (a) Work out the value of the minibus in August 2011.

   (b) Calculate the percentage loss in value of the minibus over the three-year period.

4. Zubair invests a lump sum of 180,000 ZAR (South African rands) in a Guaranteed Savings Scheme. The annual interest rate is estimated at 4.8%.

   (a) What is the value of the investment after three years?

   (b) How long will it take for the investment to double?

   (c) If it takes n years for the investment to treble, find the value of n.

5. Stephanie is paying back a loan she took out to buy a car. She has renegotiated special payment terms on the outstanding amount, \$5950. She has arranged to pay \$300 in the first month, \$294 the next month, and so on, reducing her payment by 2% each month.

   (a) How many months will it take Stephanie to pay off the outstanding amount?

   (b) What is the value of her last monthly payment?

6. The following is an extract from a national newspaper (in April 2010) about the rise in school fees:

   'Private school fees have risen by 42% over the last five years due to inflation.'

   It is assumed that the annual rate of inflation is r% and has stayed constant over the past five years.

   (a) Find the value of r to two decimal places

   (b) Given that the fees at Donna's school were \$40,000 in April 2010, work out what Donna's school fees were in:

   (i) April 2008 (ii) April 2006.

7. Mr Abban plans to buy a new car for £25,995. He applies to the bank for a loan repayable over a period of three years at 5.9% per annum. The loan schedule is shown below, with some of the information missing. Calculate each of the three missing items on the loan schedule.

Cash Price £25,995
£2,995 Deposit
followed by 1 Payment of £651.76
and 35 monthly payments of
£..(a)......
Total Interest charged for the loan £.......(b).....
Total Amount Payable £....(c)........
Borrowing Rate 5.9% per annum

8. Jennifer bought her new BMW 3-Series car for £27,245. Her brother James bought his new Mercedes C-Class for £29,015.

The salesman claimed that:

- The BMW 3-Series is worth around 70% of its new price after three years.
- The Mercedes C-Class is worth 86% of its initial purchase price after one year.

(a) What will Jennifer's car be worth after three years, based on the salesman's claim?

(b) What is the **annual** depreciation rate of the BMW?

(c) How much of its value did James's car lose after a year, according the salesman's claim?

(d) If James sold his car three years later for £20,000, what was the equivalent **annual** rate of depreciation?

(e) Based on the salesman's claim, was James better off selling his car or not? Explain your answer.

Past paper questions

1. On Vera's 18th birthday she was given an allowance from her parents. She was given the following choices.

- Choice A: \$100 every month of the year
- Choice B: a fixed amount of \$1100 at the beginning of the year, to be invested at an interest rate of 12% per annum, compounded monthly
- Choice C: \$75 the first month and an increase of \$5 every month thereafter
- Choice D: \$80 the first month and an increase of 5% every month thereafter

(a) Assuming that Vera does not spend any of her allowance during the year, calculate, for each of the choices, how much money she would have at the end of the year. *[8 marks]*

(b) Which of the choices do you think that Vera should choose? Give a reason for your answer. *[2 marks]*

(c) On her 19th birthday, Vera invests \$1200 in a bank that pays interest at r% per annum compounded annually. Vera would like to buy a scooter costing \$1452 on her 21st birthday. What rate will the bank have to offer her to enable her to buy the scooter? *[4 marks]*

[Total 14 marks]

[Nov 2002, Paper 2, Question 3] (© *IB Organization 2002*)

2. Annie is starting her first job. She will earn a salary of \$26,000 in the first year and her salary will increase by 3% every year.

(a) Calculate how much Annie will earn in her 5th year of work. *[3 marks]*

Annie spends \$24,800 of her earnings in her first year of work. For the next few years, inflation will cause Annie's living expenses to rise by 5% per year.

(b) (i) Calculate the number of years it will be before Annie is spending more than she earns.

(ii) By how much will Annie's spending be greater than her earnings in that year? *[6 marks]*

[Total 9 marks]

[May 2006, Paper 2, Question 4(ii)] (© *IB Organization 2006*)

3. *Give all answers in this question correct to the* ***nearest*** *dollar.*

Clara wants to buy some land. She can choose between two different payment options.

Both options require her to pay for the land in **20** monthly instalments.

- Option 1: The first instalment is \$2500. Each instalment is \$200 more than the one before.
- Option 2: The first instalment is \$2000. Each instalment is 8% more than the one before.

(a) If Clara chooses option 1,

(i) write down the values of the second and third instalments;

(ii) calculate the value of the final instalment;

(iii) show that the **total amount** that Clara would pay for the land is \$88,000. *[7 marks]*

(b) If Clara chooses option 2,

(i) find the value of the second instalment;

(ii) show that the value of the fifth instalment is \$2721. *[4 marks]*

(c) The price of the land is \$80,000. In option 1 her total repayments are \$88,000 over the 20 months. Find the annual rate of simple interest which gives this total. *[4 marks]*

(d) Clara knows that the **total amount** she would pay for the land is not the same for both options. She wants to spend the least amount of money. Find how much she will save by choosing the cheaper option. *[4 marks]*

[Total 19 marks]

[May 2008, Paper 2, Question 10] (© *IB Organization 2008*)

4. Sven is travelling to Europe. He withdraws \$800 from his savings and converts it to euros. The local bank is buying euros at \$1 : €0.785 and selling euros at \$1 : €0.766.

   (a) Use the appropriate rate above to calculate the amount of euros Sven will receive.

   (b) Suppose the trip is cancelled. How much will he receive if the euros in part (a) are changed back to dollars?

   (c) How much has Sven lost after the two transactions? Express your answer as a percentage of Sven's original \$800.

[May 2006, Paper 1, Question 10] (© *IB Organization 2006*)

5. Emma places €8000 in a bank account that pays a nominal interest rate of 5% per annum, compounded quarterly.

   (a) Calculate the amount of money that Emma would have in her account after 15 years. Give your answer correct to the nearest euro. *[3 marks]*

   (b) After a period of time she decides to withdraw the money from this bank. There is €9058.17 in her account. Find the number of months that Emma had left her money in the account. *[3 marks]*

[May 2008, Paper 1, Question 8] (© *IB Organization 2008*)