3 Logic, sets and probability

1899: the United States patents office predicts that, 'Everything that can be invented, has been invented.' (A misquote attributed to Charles Holland Duell, Commissioner of the US Patent Office, 1899.)

1943: the Chairman of IBM predicts that there will, 'only be a world market for maybe five computers.' (Commonly attributed to Thomas J. Watson, Chairman of International Business Machines (IBM), 1943.)

1992: spare capacity on cell phones is used to send messages in text. It is predicted to be useful as a portable paging system for people who use their cars as an office, but for no-one else.

These examples suggest that we are not very good at predicting the future, and yet forecasting and prediction is now a central part of everyone's lives.

Who uses probability?

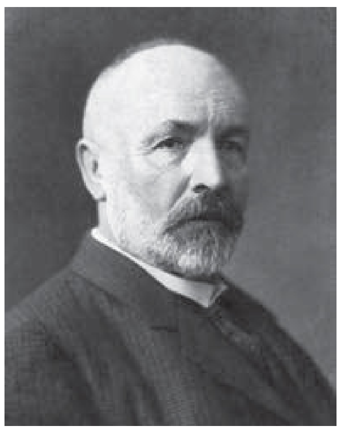
- Every time that you use something new, take a prescribed tablet, or decide what you are going to do tomorrow, your decision may have been made based on probabilities.
- Will your hockey match take place tomorrow? It depends on the weather forecast.
- What style of T-shirts will be fashionable next year? It depends on the trend forecasters and marketing teams of big corporations.

'Perfect accuracy (in forecasting) is not obtainable', say Richard Brealey and Stewart Myers in the journal *Finance*. 'But the need for planning in business is vital.' (*Source*: Richard Brealey and Stewart Myers, *Principles of Corporate Finance* (McGraw Hill, 1988).) Businesses must utilise accurate forecasting methods.

Chapter 8 Set theory and Venn diagrams

Organising **sets** or groups of objects that share characteristics is a way of remembering them and of understanding them. To do this, mathematicians use set theory to describe different groups and **Venn diagrams** to illustrate the sets. Venn diagrams are useful for practical problems and for numerical ones.

Set theory was developed by Georg Cantor (1845–1918) and revolutionised almost every mathematical field that was being studied at that time. Although his ideas were initially regarded as controversial and contentious, they have since been universally recognised for their importance and their impact on the study of mathematics.



Georg Cantor.

Venn diagrams were developed by the English mathematician John Venn (1834–1923). He taught logic and probability theory at Cambridge University and developed a method of using intersecting circles to illustrate and explain his ideas to students. In his later career he wrote a book called 'The Logic of Chance' that was influential in the study of statistical theory. He was also very skilled at building machines, including one that bowled cricket balls.

In this chapter you will learn:

- how to use the notation of set theory including elements, intersection, union, complement and subsets
- how to construct and interpret Venn diagrams with simple applications
- how to solve problems using Venn diagrams.

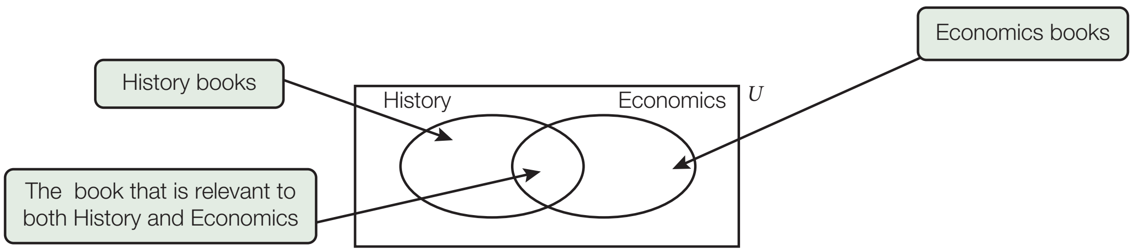
Venn diagrams are used in many contexts. Many people use them instinctively when they draw a picture to illustrate a problem that they have to solve. For instance, a director of Human Resources could use the picture on the left; do they have two teams of specialists in different areas, but only one person has expertise in both areas and can therefore move between the two teams? Is there any other combination of personnel that the department can use?

8.1 Basic concepts of set theory

How do you organise your study books – by subject, by size, by colour or by weight? Do you have any books that can be used in more than one subject? Where do you put those books?

Johan organises his books by subject; he has three on History, three on Economics and one that is relevant to both Economics and History.

A Venn diagram can be used to illustrate the overlap:



Venn diagrams are always enclosed in a rectangle. This rectangle is called the **universal set** and defines the numbers or objects that you are considering. The letter ***U*** next to the rectangle indicates that this is a universal set.

When written out, sets are always contained within curly brackets. Two or more sets are identical if they contain the same items or elements in any order. For example, $\{H_2, H_1, H_3\} = \{H_3, H_1, H_2\} = \{H_1, H_2, H_3\}$.

In the set of Johan's books:

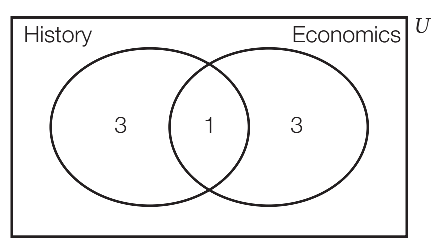
the number of books relevant to History, $n(H) = 4$,

the number of books relevant to Economics, $n(E) = 4$,

the number of books relevant to History and Economics, $n(H \text{ and } E) = 1$,

and the number of books relevant to History or Economics or both, $n(H \text{ or } E) = 7$.

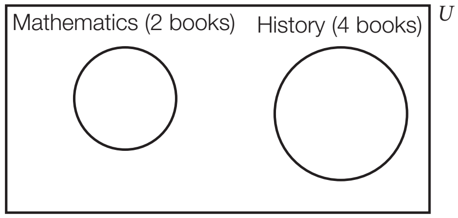
The Venn diagram will now look like this:



The last two statements can be written as:

$n(H \text{ and } E) = n(H \cap E) = 1$	The symbol for 'and' is $\cap$ and is called the **intersection**.
$n(H \text{ or } E) = (H \cup E) =$ $3 + 3 + 1 = 7$	The symbol for 'or' is $\cup$ and is called the **union**.

Johan also has two Mathematics books but there is no overlap between these and his History books so the Venn diagram for Johan's books relevant to History and to Mathematics looks like this:



From the diagram you can see that the sets are separate; there is no overlap. So:

the number of History books, $n(H) = 4$

and the number of Mathematics books, $n(M) = 2$,

then the number of History or Mathematics books, $n(M \cup H) = 2 + 4 = 6$.

As there are no books that can be used for both Mathematics and History, $n(H \cap M) = \varnothing$.

A set with no members in it is called an **empty set** and the symbol for an empty set is $\varnothing$.

An empty set is not the same as $\{0\}$; this is a set containing the number zero so it is not empty.

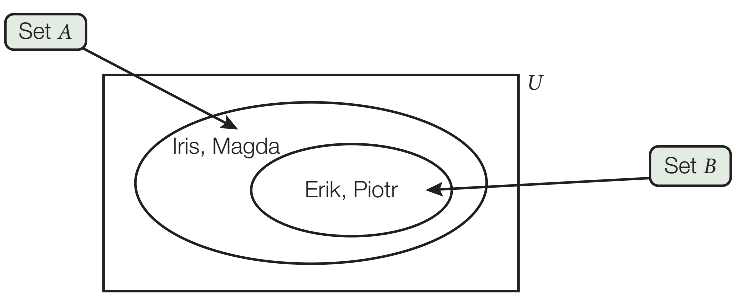
Johan's friends Magda, Iris, Erik and Piotr all play hockey. Erik and Piotr play in the same team as Johan. The universal set in this example is 'all Johan's friends'.

The Venn diagram looks like this:

$U = \{\text{all Johan's friends}\}$

$A = \{\text{Johan's friends who play hockey}\} = \{\text{M, I, E, P}\}$

$B = \{\text{friends in the same team as Johan}\} = \{\text{E, P}\}$



Notice that set B is completely enclosed by set A. Set B is called a **subset** of set A; all the members of B are also in A.

You can use some extra notation:

$\text{M} \in A$ means that Magda is a member, or element, of set A.

$\text{I} \notin B$ means that Iris is not a member of set B, as she is not in the same hockey team as Johan.

$B \subset A$ means that B is a subset of A. Every element of B is enclosed within set A.

An empty set is also a subset of all other sets.

Summary of notation

{}	curly brackets	lists the members of a set
U	universal set	defines the field being considered
$n(A)$		number of elements in a set A
$\in$	element	a member of a set
$\notin$	not an element	not a member of a set
$\varnothing$	empty set	a set with no elements or members
$\cap$	intersection	$A \cap B$ is the overlap of set A and set B (A and B)
$\cup$	union	$A \cup B$ contains all the elements of A and of B (but without any repeats) (A or B)
$\subset$	subset	a set that is equal to or enclosed by another set

exam tip

In examinations, questions are unlikely to contain more than two or three subsets in any universal set.

Exercise 8.1

1. Write the following statements, using set notation:

(a) x is a member of set A

(b) x is not a member of set A

(c) B is a subset of C

(d) C union D

(e) A intersection B.

2. Write the following statements, using set notation:

(a) the elements of set A are x, y and z

(b) the number of elements in both sets A and B is 3

(c) set B consists of the vowels, a, e, i, o, u

(d) the number of elements in set A is 5.

8.2 Venn diagrams with numbers

Venn diagrams are a good way of organising sets of numbers so that you can see the links between them.

Let $U = \{1, 2, 3, \ldots, 9, 10\}$	All the integers from 1 to 10.
$A = \{2, 4, 6, 8, 10\}$	All the even natural numbers between 1 and 10 inclusive, i.e. including 10.
$B = \{2, 3, 5, 7\}$	All the prime numbers between 1 and 10.

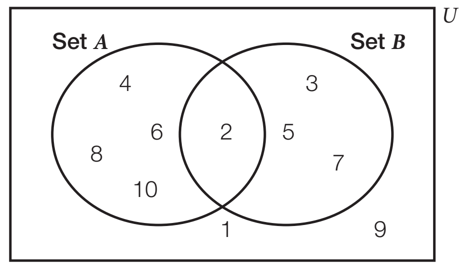
Before you fill in the Venn diagram, think about the following:

2 is in both set A and set B, so goes in the overlap of A and B.

1 and 9 are in neither set A nor set B, so they go outside the circles.

3, 5 and 7 are prime but not even, so they go inside set B but outside set A.

6, 4, 8 and 10 are even but not prime numbers, so they go inside set A but outside set B.



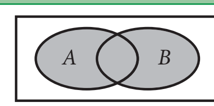
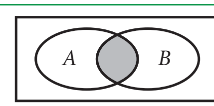
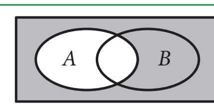
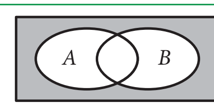
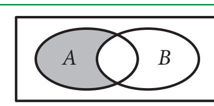
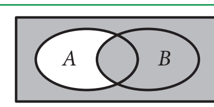
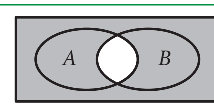
Now the numbers are organised, the diagram shows that:

$n(U) = 10$
$n(A) = 5$
$n(B) = 4$

$5 \notin$ set A	(Remember $\notin$ means 'is not a member'.)
$9 \notin$ set A or set B	
$2 \in$ set A and set B	(Remember $\in$ means 'is a member'.)
$A' = \{1, 3, 5, 7, 9\}$	A' means the **complement** of A, which consists of all the numbers that are not in set A.

You can see that set theory can be expressed in **notation**, in **set language** or by using **diagrams**.

The table below shows the notation and explains it using diagrams, based on the example above.

Set notation	Set language	Meaning	Venn diagram	Answer
$A \cup B$	A union B.	Everything that is in either or both sets.		$\{2, 3, 4, 5, 6, 7, 8, 10\}$
$A \cap B$	A intersection B.	Everything that is in the overlap of both sets.		$\{2\}$
A'	The complement of A.	Everything that is not in set A.		$\{1, 3, 5, 7, 9\}$
$(A \cup B)'$	The complement of $(A \cup B)$.	Everything that is not in set A or set B.		$\{1, 9\}$
$A \cap B'$	The intersection of A and the complement of B.	The overlap between A and everything that is not in B.		$\{4, 6, 8, 10\}$
$A' \cup B$	The union of B with the complement of A.	Everything that is in B or not in A.		$\{1, 2, 3, 5, 7, 9\}$
$(A \cap B)'$	The complement of A intersection B.	Everything that is not in the overlap of A and B.		$\{1, 3, 4, 5, 6, 7, 8, 9, 10\}$

In Chapter 1, the definitions of numbers are given using set language. Natural numbers were defined as $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$.

Using this notation you can see that, for this course, natural numbers include zero and all the positive counting numbers from one to infinity.

If integers are defined as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$, then you can see the difference between the definitions very quickly.

Worked example 8.1

Q. Let $U = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ (all the integers from -5 to $+5$),

$A = \{-5, -4, -3, -2, -1\}$ (negative integers from -5 to $+5$),

$B = \{-4, -2, 0, 2, 4\}$(even integers from -5 to $+5$).

(a) Draw the Venn diagram.

(b) Use the diagram to answer the following questions.

(i) Is $A \cap B$ an empty set?

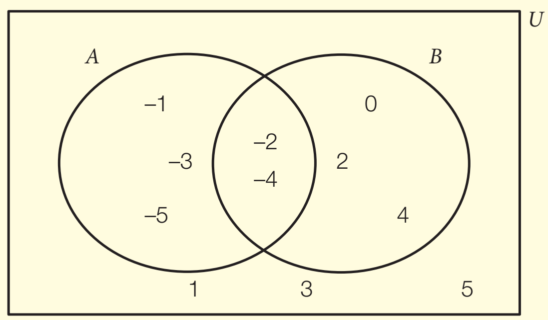
(ii) List the elements of $A \cup B$.

(iii) List the elements of $A \cap B'$.

(iv) Find $n(A)$, $n(A \cap B)$ and $n((A \cup B)')$.

(v) Describe the set $A \cap B$ in words.

A. (a)



(b) (i) $A \cap B \neq \varnothing$

'*A* intersection *B*' is not an empty set, because it contains the numbers –2 and –4.

(ii) $A \cup B = \{-5, -4, -3, -2, -1, 0, 2, 4\}$

'*A* union *B*' consists of all the numbers that are in *A* and/or *B*.

(iii) $A \cap B' = \{-1, -3, -5\}$

All the numbers that are in *A* and not in *B*.

(iv) $n(A) = 5$

$n(A \cap B) = 2$

$n((A \cup B)') = 3$

All those numbers that are not in A or B.

(v) $A \cap B$ contains numbers that are both negative and even.

Exercise 8.2

1. (a) If $P = \{1, 4, 8, 12, 20, 32, 52, 84\}$, state $n(P)$.

   (b) If $Q = \{$square numbers less than 40$\}$, state $n(Q)$.

   (c) If $R = ($prime numbers between zero and 30$\}$, state $n(R)$.

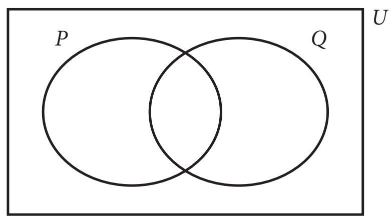
2. Two sets of real numbers A and B are defined as follows:

   $A = \{9, 10, 11, 12, 13, 14, 15\}$

   $B = \{5, 6, 7, 8, 9, 10, 11\}$

   List the elements of sets $A \cup B$ and $A \cap B$.

3. Copy this Venn diagram three times.



   (a) On one copy, shade the area that represents $P \cap Q$.

   (b) On another copy, shade the area that represents $(P \cap Q)'$.

   (c) On the third copy, shade the area that represents the complement of $(P \cup Q)$.

4. The universal set U is the set of integers from 1 to 64 inclusive. A and B are subsets of U such that:

   A is the set of square numbers,

   B is the set of cubed numbers.

   List the elements of the following sets:

   (a) A (b) B (c) $A \cup B$ (d) $A \cap B'$.

8.3 Applications of set theory and Venn diagrams

Venn diagrams can be useful when solving practical problems.

Worked example 8.2

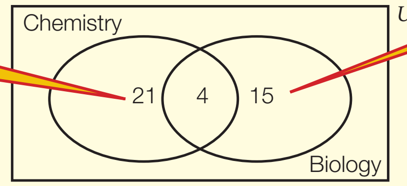
Q. There are 40 students at an IB school who are studying either Chemistry, Biology or both. 25 students are studying Chemistry and 19 students are studying Biology. 4 students are studying both subjects. How many students study just Chemistry or just Biology?

If people work in international teams and have to communicate quickly, diagrams can be a quick way of ensuring that everyone understands an idea, even if they are not working in their first language. Some people remember ideas better if they see them as a picture too.

continued . . .

A. Let $U = \{$students studying Chemistry and/or Biology$\}$
$= 40$

25 students, but 4 are studying Biology too.
$25 - 4 = 21$



$19 - 4 = 15$

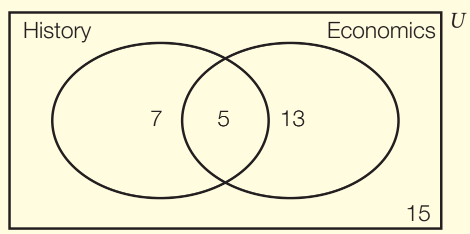
21 students study just Chemistry.

15 students study just Biology.

Worked example 8.3

Q. Of the same group of 40 students, 12 take History, 18 take Economics and 5 take both subjects. How many students do not take History or Economics?

A. Let $U = \{$students studying History and/or Economics$\}$



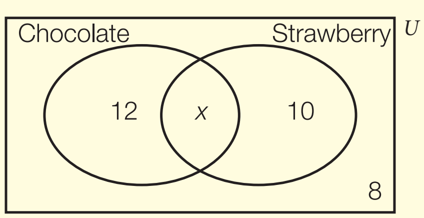
$7 + 5 + 13 = 25$
$40 - 25 = 15$
There are 15 students who do not take History or Economics.

There are 15 students who do not take History or Economics.

Worked example 8.4

Q. There are 32 students at a party. 12 students say that they only like chocolate-pecan ice cream and 10 students say that they only like strawberry-and-cookies ice cream. 8 students do not like either. How many students like both?

A.



There are 32 students, so

$$12 + x + 10 + 8 = 32$$
$$x + 30 = 32$$
$$x = 2$$

Two students like both chocolate-pecan and strawberry-and-cookies ice cream.

Exercise 8.3

1. The universal set U is defined as:

   $U = \{x \in \mathbb{Z}: 41 \le x \le 50\} \equiv$ {all the integers from 41 to 50 inclusive}

   Subsets X and Y of U are defined as:

   $X =$ {multiples of 6}

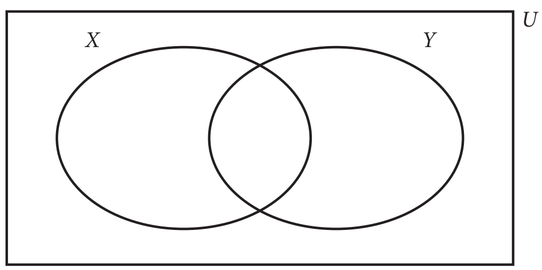
   $Y =$ {multiples of 7}.

   (a) List the elements of:

   (i) $X \cap Y$ (ii) $X \cap Y'$.

   (b) Find $n((X \cap Y)')$.

   (c) Illustrate the information from (a) on copies of this Venn diagram.



hint

Here is a reminder of what these symbols mean:
≥ greater than or equal to;
≤ less than or equal to;
≡ equivalent to.

2. Let U be the set of all positive integers from 5 to 55 inclusive.

   A, B and C are subsets of U such that:

   A is the set of prime numbers contained in U,

   B is the set of multiples of 11 contained in U,

   C contains all the positive integers that are factors of 55.

   (a) List all the members of set A.

   (b) Write down all the members of:

   (i) $B \cup C$ (ii) $A \cap B \cap C$.

3. The universal set U is defined as all positive integers between 11 and 43 inclusive.

   A, B and C are subsets of U such that:

   $A =$ {factors of 36}, $B =$ {multiples of 4} and $C =$ {multiples of 6}.

   (a) Find $n(B)$.

   (b) List the elements in $A \cap B \cap C$.

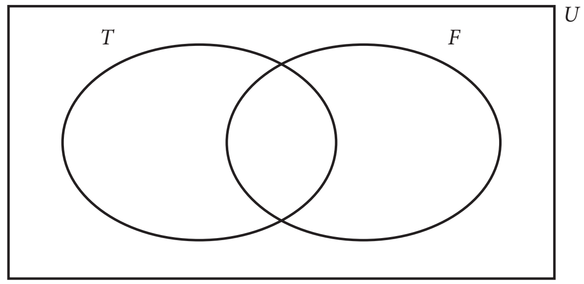
   (c) List the elements in $A \cap B'$.

4. During a school's Sports Day activities, students participated in Track and Field events.

   U = {Track and Field events}

   T = {Track events, mainly running}

   F = {Field events, mainly jumping, throwing, etc.}

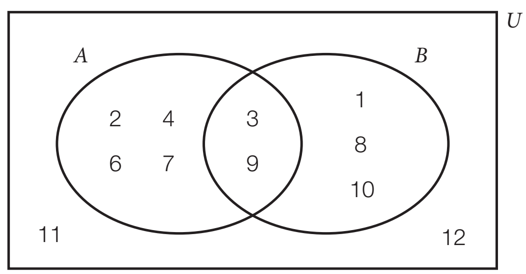


   On three separate copies of the Venn diagram from above, shade the following regions. Write in words what each region represents.

   (a) $T \cap F'$ (b) $(T \cap F)'$

5. In the Venn diagram, sets A and B are subsets of the universal set U.

   U is defined as the positive integers between 1 and 12 inclusive.



   (a) Find:

   (i) $n(A)$ (ii) $n(A \cap B)$.

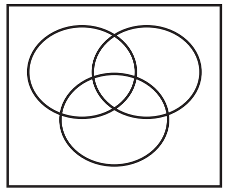
   (b) List the elements in:

   (i) $(A \cup B)'$ (ii) $A' \cap B$.

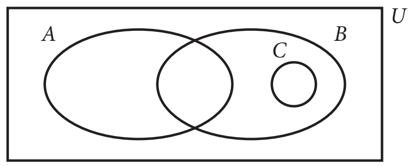
8.4 Venn diagrams with three sets

Venn diagrams can be used to solve problems with three sets. The notation and definitions for three sets are the same as those with two sets, but the diagrams are more complicated.

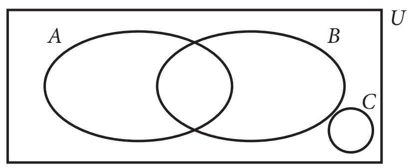
Most problems with three sets give you a diagram like this:



But you need to read the questions carefully, because you may get one like this:



Or like this:



In some problems the Venn diagram has already been completed, and can be used to answer questions.

Is it possible to draw meaningful Venn diagrams with four or more sets? The first task is to define the different sets with common features that can intersect. Then you need to ensure that the diagram clearly illustrates all aspects of the initial problem. How do you fit a fourth set into the diagram for three sets, and keep the meaning clear?

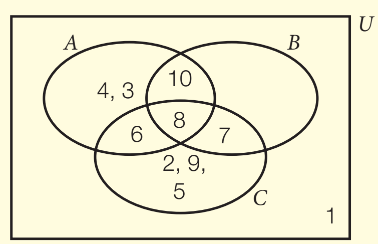
Worked example 8.5

Q. In this diagram the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{3, 4, 6, 8, 10\}$

$B = \{7, 8, 10\}$

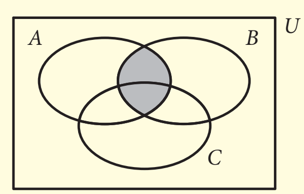
$C = \{2, 5, 6, 7, 8, 9\}$



Use the diagram to list the elements in the following sets and shade their position on the Venn diagram:

A. (a) $A \cap B = \{8, 10\}$

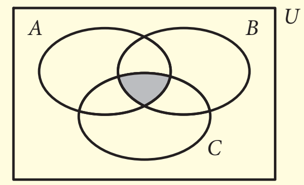
(a)



continued . . .

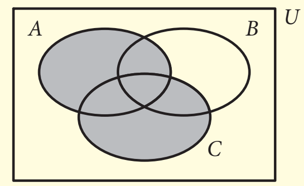
(b) $A \cap B \cap C = \{8\}$

(b)



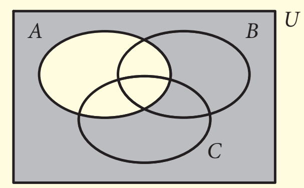
(c) $A \cup C = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(c)



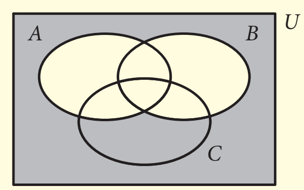
(d) $A' = \{1, 2, 5, 7, 9\}$

(d)



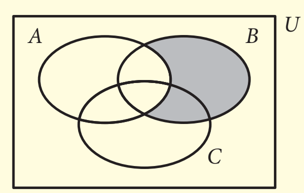
(e) $(A \cup B)' = \{1, 2, 5, 9\}$

(e)



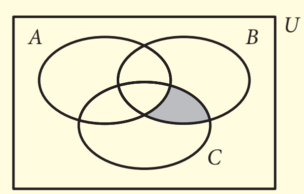
(f) $A' \cap B = \{7\}$

(f)



(g) $(A' \cap B) \cap C = \{7\}$

(g)



Worked example 8.6

Q. There are 50 people in a band. The conductor wants to know who can play the saxophone, who can play the trumpet and who can play the drums.

He discovers that:

2 people can play all three instruments,

5 people can play the saxophone and the trumpet,

1 person can play the saxophone and the drums,

3 people can play the trumpet and the drums,

22 people can play the trumpet,

15 people can play the drums,

18 people can play the saxophone.

How many people cannot play any of these instruments?

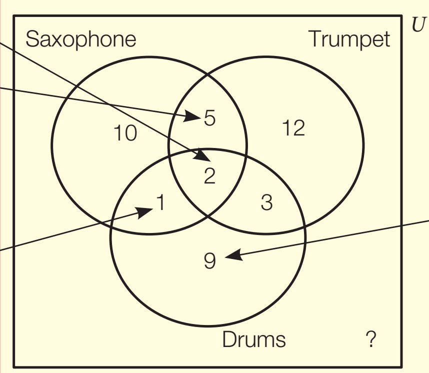
To complete the Venn diagram you should start in the middle and work outwards. In this example the information has been given in the correct working order, but many questions may not do this. So look for the information that goes in the centre first!

A.

Start with two people in the centre, $n(S \cap T \cap D) = 2$

Then look at the other intersections: 5 people play saxophone and trumpet, so put a 5 in the overlap of just S and T.

1 person plays saxophone and drums, so put a 1 in the overlap of just S and D; similarly for the 3 people who play trumpet and drums.



15 people play the drums but $1 + 2 + 3 = 6$ have already been included in the intersections with saxophone and trumpet, leaving 9. Do the same to get 10 in S only and 12 in T only.

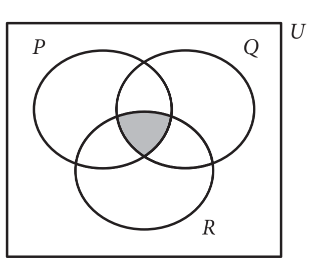
There are 42 people in the band who play saxophone, trumpet or drums.

There are $50 - 42 = 8$ people who do not play any of these instruments.

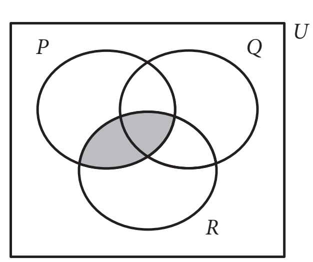
Exercise 8.4

1. Use set notation to represent the shaded region in the following Venn diagrams:

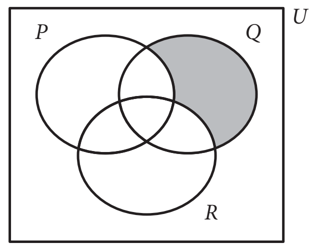
(a)



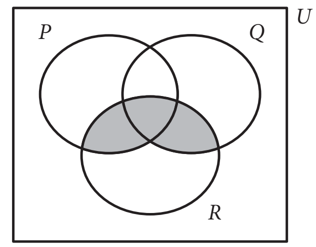
(b)



(c)



(d)



2. The universal set U is defined as $U = \{a, c, e, f, g, h, j, k\}$.

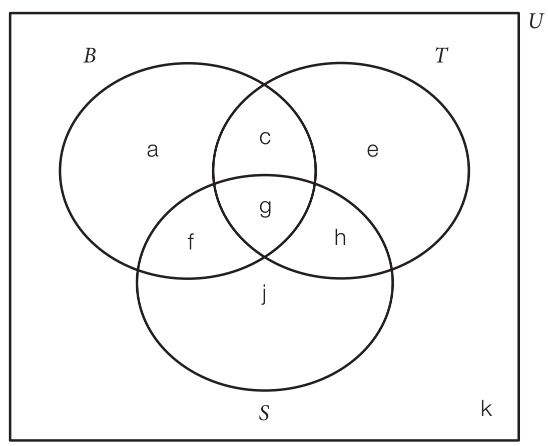
B, T and S are subsets of U such that:

$B = \{a, c, f, g\}$

$T = \{c, e, g, h\}$

$S = \{f, g, h, j\}$.

The information from above is illustrated on the Venn diagram:



From the Venn diagram, list the elements of each of the following regions:

(a) $B \cap T \cap S$

(b) $(B \cap T)'$

(c) $(B \cap T)' \cap S$

(d) $B \cap (T \cup S)$.

3. 60 teachers were asked which sports programmes they watched on TV during one evening after school.

18 watched Athletics (A),

27 watched Cricket (C),

20 watched Soccer (S),

3 watched all three sports programmes,

1 watched Cricket and Soccer only,

4 watched Athletics and Soccer only,

5 watched Athletics and Cricket only.

(a) Draw a Venn diagram to illustrate the relationship between the sports programmes.

(b) On your diagram indicate the number of teachers belonging to each region.

(c) Determine the number of teachers who did not watch any of the three sports programmes mentioned above.

4. 72 students were asked which subject they revised over the weekend.

21 revised Mathematics (M) only,

8 revised French (F) only,

11 revised Physics (P) only,

2 revised all three subjects,

3 revised Mathematics and French,

5 revised French and Physics,

7 revised Mathematics and Physics.

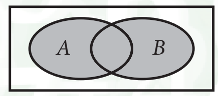
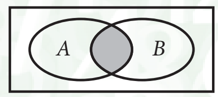
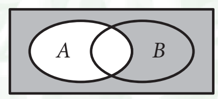
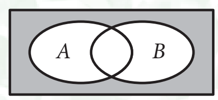
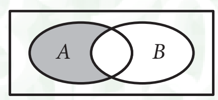
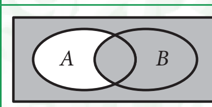
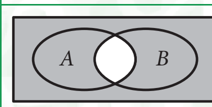
(a) Represent the above information on a Venn diagram, indicating the number of students belonging to each region.

(b) How many students revised neither Mathematics nor French nor Physics?

Summary

You should know:

- the foundations of set theory including what is meant by elements, intersection, union, complement and subsets
- the different ways of expressing set theory: set notation, set language and Venn diagrams

Set notation	Set language	Meaning	Venn diagram
$A \cup B$	A union B.	Everything that is in either or both sets.	
$A \cap B$	A intersection B.	Everything that is in the overlap of both sets.	
A'	The complement of A.	Everything that is not in set A.	
$(A \cup B)'$	The complement of $(A \cup B)$.	Everything that is not in set A or set B.	
$A \cap B'$	The intersection of A and the complement of B.	The overlap between A and everything that is not in B.	
$A' \cup B$	The union of B with the complement of A.	Everything that is in B or not in A.	
$(A \cap B)'$	The complement of A intersection B.	Everything that is not in the overlap of A and B.	

- about Venn diagrams with simple applications and problems that can be solved by using them.

Mixed examination practice

Exam-style questions

1. The following data represents the results from a survey of students in the same tutor group:

U = {students in tutor group}, $n(U) = 50$

H = {History students}, G = {Geography students}

$n(H) = 16$, $n(G) = 24$, $n(H \cap G) = 2$.

(a) Explain in words what the region $(H \cup G)'$ represents.

(b) State the value of $n(H \cap G)'$.

(c) Draw a Venn diagram to represent the data from above, indicating the number of students in each region of the diagram.

2. A survey was carried out on 40 students about when they did their homework over the weekend. The results are shown below:

F = {homework done on Friday night}

S = {homework done on Saturday/Sunday}

$n(F) = 20$, $n(S) = 29$.

(a) State the value of $n(F \cap S)$.

(b) What does $(F \cup S)'$ mean? Explain why $n(F \cup S)' = 0$.

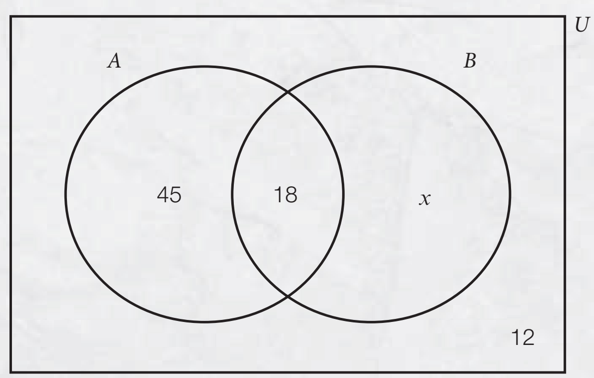
(c) Draw a Venn diagram to represent the data from above, indicating the number of students in each region of the diagram.

3. 100 Mathematics teachers who attended a conference were asked which programme of study they had taught in the last 18 months. Their responses are illustrated on the Venn diagram below:

U = {Mathematics teachers at conference}

A = {teachers who have taught on the Advanced Level programme}

B = {teachers who have taught on the IB Diploma programme}



(a) Describe the region denoted by x using:

(i) words (ii) set notation.

(b) State the value of x.

(c) Find: (i) $n((A \cap B)')$ (ii) $n((A \cup B)')$.

4. A survey which asked 120 people about what types of books and materials they read most recently in their local library provided the following results:

71 read fiction books (F),

54 read non-fiction books (N), including textbooks,

44 read reference books (R), including journals and newspapers,

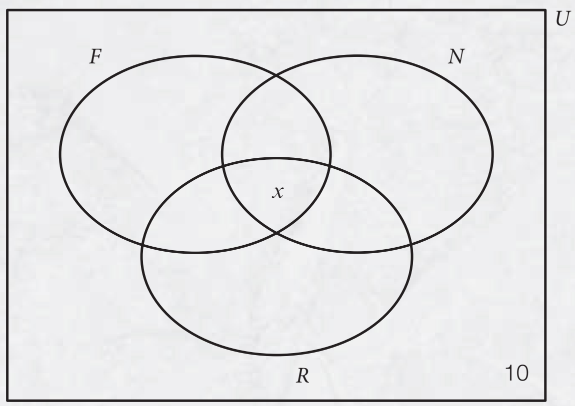
20 read both fiction and reference,

8 read reference and non-fiction but not fiction,

15 read fiction and non-fiction but not reference,

x people read all three types of materials,

10 read none of the types of materials mentioned above.



(a) Show that $n(F \cap R \cap N') = 20 - x$.

(b) Find, in terms of x, $n(N \cap R' \cap F')$.

(c) Complete the Venn diagram, indicating the number of people corresponding to each region.

(d) Hence, or otherwise, find the value of x.

5. 100 students were asked which resources they used during their revision for their final examination in Mathematics. The three main resources were:

C = {CDs, videos, etc.}

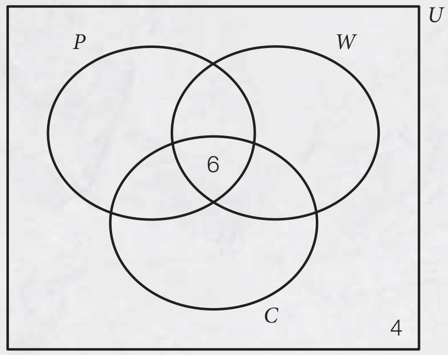
P = {printed materials, including textbooks, etc.}

W = {web/internet resources}

The number of students representing the corresponding regions are:

$n(P) = 51$	$n(C \cap W) = 13$
$n(W) = 32$	$n(C \cap P \cap W) = 6$
$n(C) = 63$	$n[(C \cup P)' \cap W] = 6$
$n(P \cap C) = 24$	$n(C \cup P \cup W)' = 4$

Complete the Venn diagram below using the information given above.

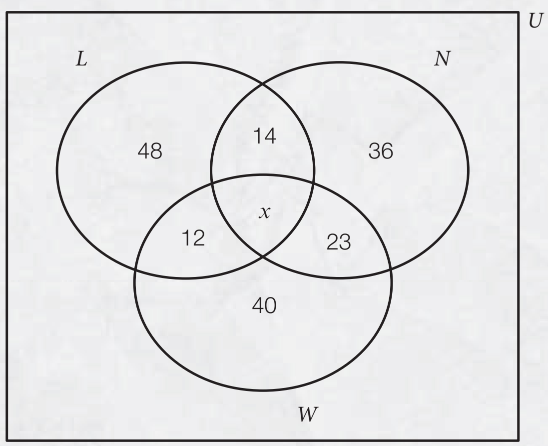


6. The principals of 180 colleges were asked where they advertised for teachers to fill vacant positions. Their responses are illustrated on the Venn diagram below, where:

L = {local newspapers}

N = {national newspapers}

W = {web/internet}



(a) Given that $n(L \cup N \cup W)' = 0$, determine the value of x.

(b) From the information in the Venn diagram, write down the number of principals who advertised:

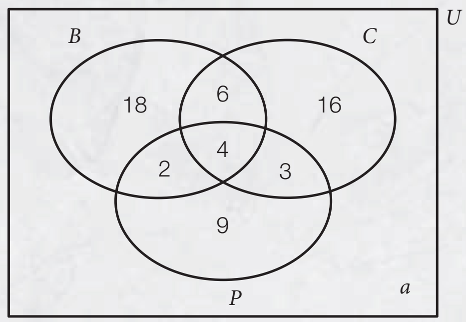
(i) in both local newspapers and on the internet

(ii) in local newspapers and/or on the internet

(iii) in both national newspapers and on the internet but not in local newspapers

(iv) in local and/or national newspapers but not on the internet.

7. The following Venn diagram shows the number of students who study Biology (B), Chemistry (C) and Physics (P) in a college.



(a) Find:

(i) $n(B \cap C \cap P)$ (ii) $n(C \cup P)$.

Given that $n(U) = 100$, find:

(b) the value of a

(c) $n(B')$.

8. The Mathematics Enrichment Club in a school runs sessions on Mondays (M), Wednesdays (W) and Fridays (F). A number of students were asked which of the enrichment sessions they had attended during the previous week.

8 students attended on Monday only,

6 students attended on Monday and Wednesday but not on Friday,

7 students attended on Monday and Friday but not on Wednesday,

3 students attended on Wednesday and Friday but not on Monday,

20 students did not attend any of the sessions.

(a) Illustrate the above information on a Venn diagram.

Given that:

25 students attended on Monday,

24 students attended on Wednesday,

35 students attended on Friday,

(b) find the number of students who attended all three sessions during the week

(c) find the total number of students in the group

(d) hence complete the Venn diagram from part (a).

9. 300 tourists were asked which attractions they had seen while in London.

Most of them had seen Buckingham Palace (B), Trafalgar Square (T) and Westminster Abbey (W).

25 people had seen all three attractions,

52 people had seen both Trafalgar Square and Westminster Abbey,

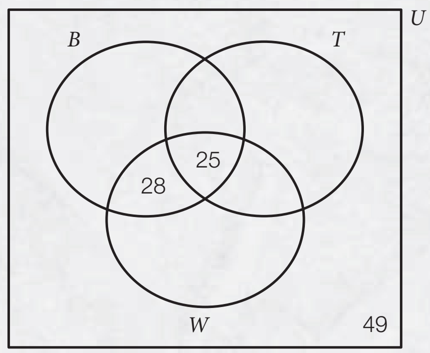
28 people had seen Buckingham Palace and Westminster Abbey but not Trafalgar Square,

88 people had seen exactly two of the three attractions,

211 people had seen Buckingham Palace or Trafalgar Square,

199 people had seen Trafalgar Square or Westminster Abbey,

49 people had seen other attractions, but none of the three places listed above.



Use the information from above to complete the Venn diagram, indicating the number of people representing each of the regions.

10. 120 customers in a music shop were asked about the genre of music they had just bought from the shop. The three main genres were Classical (C), Folk (F) and Pop (P).

4 customers bought all three genres of music,

60 customers bought Pop or Folk music but not Classical,

59 customers bought Pop music,

30 customers bought at least two of the three genres of music,

7 customers bought Pop and Classical but not Folk music,

10 customers bought both Classical and Folk music,

25 customers bought none of these three genres of music.

Illustrate the information from above on a Venn diagram, indicating the number of customers for each region.

Past paper questions

1. At a certain school there are 90 students studying for their IB diploma. They are required to study at **least one** of the subjects: Physics, Biology or Chemistry.

50 students are studying Physics,

60 students are studying Biology,

55 students are studying Chemistry,

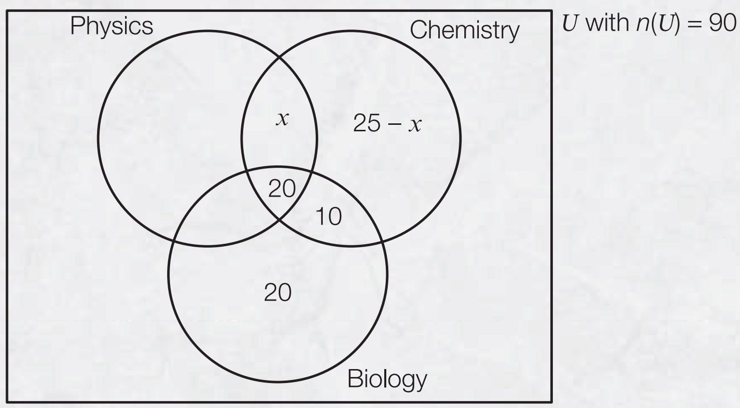
30 students are studying both Physics and Biology,

10 students are studying both Biology and Chemistry but not Physics,

20 students are studying all three subjects.

Let x represent the number of students who study both Physics and Chemistry but not Biology. Then $25 - x$ is the number who study Chemistry only.

The figure below shows some of this information and can be used for working.



(a) Express the number of students who study Physics only, in terms of x.

(b) Find x.

(c) Determine the number of students studying **at least two** of the subjects.

[Total 6 marks]

[May 2006, Paper 1, Question 13] (© *IB Organization 2006*)

2. A school offers three activities, basketball (B), choir (C) and drama (D). Every student must participate in at least one activity.

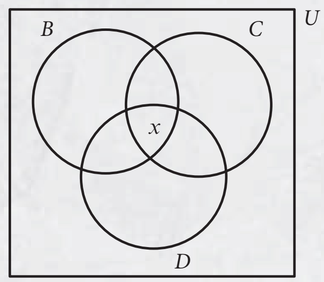
16 students play basketball only,

18 students play basketball and sing in the choir but do not do drama,

34 students play basketball and do drama but do not sing in the choir,

27 students are in the choir and do drama but do not play basketball,

(a) Enter the above information on the Venn diagram below. *[2 marks]*



99 of the students play basketball, 88 sing in the choir and 110 do drama.

(b) Calculate the number of students x participating in all three activities. *[1mark]*

(c) Calculate the total number of students in the school. *[3 marks]*

[Total 6 marks]

[**Nov 2007, Paper 1, Question 8**] (© *IB Organization 2007*)

Chapter 9 Logic

In this chapter you will learn:

- about the principles of symbolic logic
- how propositions are defined and their notation
- about compound statements, their truth tables and Venn diagrams
- how to use truth tables to test arguments, and the concepts of logical equivalence, contradiction and tautology
- how converse, inverse and contrapositive statements are constructed.

William Jevons' logic piano.

The logic piano was built for William Jevons (1835–1882) and is now in the Museum of the History of Science in Oxford, England. It was designed to create **truth tables** for up to four **propositions** at a time. William Jevons' friend, John Venn, could see no practical use for it as he could think of no set of propositions that would need a mechanical device to interpret them. Nevertheless, Jevons used the logic piano in both his teaching and his personal studies.

But what is logic? What are truth tables? Logic is described as the science of thinking, of reasoning, and of proof. It should not be confused with the everyday meaning of 'logical thinking' that is commonly used by people to describe a sensible route of thinking, or 'common sense'; here it is a specific mathematical process. It is the careful study of the patterns of arguments, particularly those that start with a true **statement** and go on to a valid conclusion. Truth tables are diagrams that help to interpret and draw conclusions from the logical process.

The study of logic began centuries ago. In India, Medhatithi Gautama (*c*. 6th century BCE) developed logic for religious and philosophical arguments. In Greece, the philosopher Aristotle introduced the concept of syllogisms. In China, the Mohist school encouraged an interest in logic and the solving of logical puzzles.

This chapter is an introduction to Mathematical Logic, a topic in mathematics that is concerned with the study of formal reasoning and mathematical proof.

You will encounter logic in other fields. For example, it is part of the study of law, computing, language and politics.

One hundred years ago, A.N. Whitehead and Bertrand Russell wrote a book called *Principia Mathematica*, which attempted to express all mathematics in precise logical terms. This work led to new branches of mathematics and to the development of electronics and computing.

9.1 Propositions

Mathematical logic is based on statements, which are also known as propositions.

A **proposition** is a statement that is either true (T) or false (F); it cannot be both true and false **at the same time**. For any proposition, it is the clarity of the statement that is important.

For instance, 'All men are mortal' and 'The flower is blue' are both clear statements, and therefore propositions.

A proposition cannot be a question, an instruction or an opinion.

For example:

'The sun is shining.'	is a statement and therefore a proposition.
'Is the sun shining?'	is a question and therefore not a proposition.
'How beautiful the sun is!'	is an opinion, not a proposition.
'Tell the sun to shine.'	is an instruction, not a proposition.

hint

In determining if a statement is a proposition, it does not matter if it is a 'true' or 'false' statement; for example:

'All men are mortal' is a proposition that is always true.

'This flower is blue' is a proposition that may be true or may be false.

'The Earth is flat' is a proposition, even though it is factually incorrect.

Worked example 9.1

Q. Look at these statements and decide which ones are propositions.

Notice that mathematical statements can also be propositions.

(a) Do all cats have tails? — (a) No — This is a question.

(b) $7 > 4$ — (b) Yes — This is a statement so it is a proposition.

(c) Logic is important. — (c) No — This is an opinion.

(d) Kuala Lumpur is the capital of Malaysia. — (d) Yes — This is a statement so it is a proposition.

(e) Can cows see in colour? — (e) No — This is a question.

(f) $4.8 \notin \mathbb{R}$ — (f) Yes — This is a statement, so it is a proposition, even though it is false (4.8 is a real number).

continued . . .

(g) Mammals live on land. (g) Yes

This is a statement, so it is proposition, even though some mammals live in the sea.

(h) Our sun is a star. (h) Yes

This is a statement so it is a proposition.

(i) Write a letter. (i) No

This is an instruction.

(j) That was a good song. (j) No

This is an opinion.

Exercise 9.1

1. Consider the following statements and decide which ones are propositions. In the case of a proposition, indicate whether it is true or false.

    (a) A dollar is a unit of currency.

    (b) Come back home!

    (c) All square numbers are odd numbers.

    (d) There are 12 months in a year.

    (e) China is an Asian country.

    (f) Where is Anna?

    (g) iPods can store music.

    (h) Three little birds.

    (i) 2 is a prime number.

    (j) Is this what you are looking for?

Symbolic notation

Writing out propositions in full takes time and is often not very useful. It is easier to assign each proposition a letter, and give a single definition of that letter. The usual letters used are p, q, r, s.

For example, the proposition 'all cats have tails' can be defined by writing:

p: All cats have tails.

Similarly,

q: 4.8 is a real number.

r: Our sun is a star.

There are many other symbols in symbolic notation that are used to show the relationship between multiple propositions; you will be introduced to these as you progress through the chapter.

Negation

Negation is the opposite of the original proposition, and means 'not'. The symbol for 'not' is $\neg$. (Note that there are other symbols for 'not': $\sim p$, p' or $\overline{p}$; you might recognise $'$ from set notation.)

So, for the proposition p: It is raining,

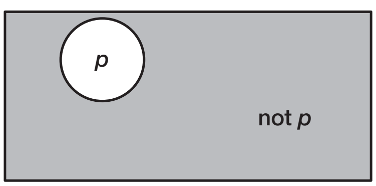
the negation is $\neg p$: It is not raining.

$\neg(\neg p)$: It is not not raining. This means the same as 'it is raining'.

You can also think of negation in terms of diagrams. The Venn diagram below shows you that everything that is false is not true. The truth table gives you another visual picture. Truth tables list each statement in symbolic notation at the top of a column in the table, and indicate if they are true (T) or false (F) in each cell of the table.

You learned about Venn diagrams in Chapter 8.

Truth tables are covered in more depth in section 9.4, page 268



p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

From the truth table you can see that:

if p is true, $\neg p$ is not true (so it is false) and therefore 'not not p' is true;

if p is false, $\neg p$ is true (because it is the opposite) and therefore 'not not p' is false.

9.2 Compound statements

Propositions can be connected to form **compound statements**.

Take the following propositions:

- It is raining.
- I have my umbrella.
- It is sunny.

You could say:

- It is raining **and** I have my umbrella.
- It is raining **or** it is sunny.
- It is **not** raining **and** it is sunny.
- It is raining **or** it is sunny, **but not both**.

Each of these is a compound statement that uses different words to connect the two propositions and produces a statement with a different meaning from the original propositions.

Notice that the key words that connect each proposition are: 'and', 'or', 'not', 'or … but not both'.

If the original proposition is a compound statement, the resulting **negation** ($\neg$) is not considered a compound statement. However, the negation is still an important part of many compound statements.

This use of Venn diagrams in a different context shows how powerful they are and how clearly they can illustrate ideas.

Conjunction, disjunction and exclusive disjunction

Conjunction, disjunction and exclusive disjunction are the technical terms for the words that connect two propositions to make a compound statement. They look complicated but the ideas they express are simple and you are already familiar with them from set theory. Each term has a symbol called a **connective** that allows you to write the whole statement.

For the following explanations of each connective, we have used a truth table and a Venn diagram to demonstrate their meaning. The truth tables illustrate the different combinations that are possible in each situation.

Conjunction

When two propositions are connected by the word 'and', the compound statement is called a **conjunction**. The logic symbol for 'and' is '$\wedge$'.

For the propositions p: It is raining

q: I have my umbrella

the compound statement 'It is raining and I have my umbrella' becomes $p \wedge q$.

p can be either true (T) or false (F).

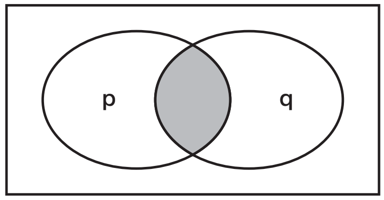
q can be either true or false.

This means there are four possible combinations that could occur with the propositions p and q (note that only one can occur at any one time):

- both propositions are true: p is true, and q is true (1)
- both propositions are false: p is false, and q is false (2)
- one proposition is true and the other false: p is true, and q is false (3)
  p is false, and q is true. (4)

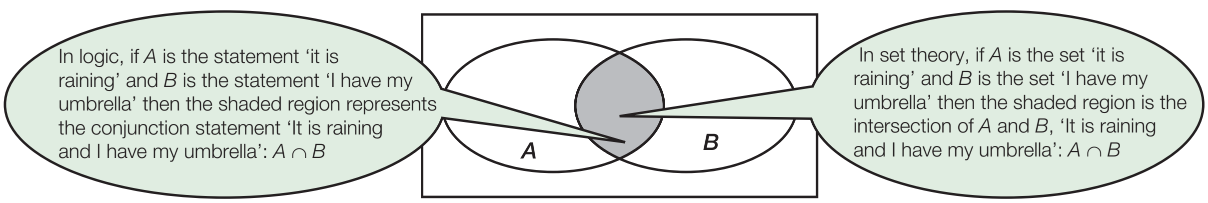
The four different situations can be illustrated much more easily using a truth table or a Venn diagram, which allow you to see under what circumstances $p \wedge q$ is true. If you start with T, T, F, F in the column 'p', then you can complete the rest of the columns accordingly.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F



You can see from the truth table and the Venn diagram that for $p \wedge q$ to be true, **both** p and q must be true. If either p or q, or both, are false, then the whole combined proposition is false.

It might help you to recognise that a conjunction in logic is similar to an intersection in set theory. If you compare the Venn diagram for a **conjunction** with the Venn diagram for the **intersection** of two sets you will see that they look the same.



You learned about intersection in set theory in Chapter 8.

In set theory, $A \cap B$ means 'the intersection of A and B' or 'A and B'.

In logic, $p \wedge q$ means 'p and q'.

Disjunction

When two propositions are connected by the word 'or', the compound statement is called a **disjunction**. The logic symbol for 'or' is '$\vee$'. A compound statement with a disjunction is true when either one, or both, of the propositions are true.

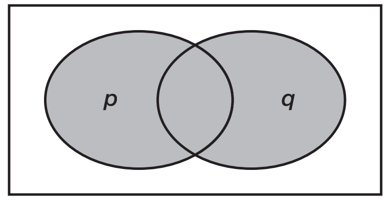
For the propositions p: It is raining

q: It is sunny

the compound statement 'It is raining or it is sunny' becomes $p \vee r$.

Both the truth table and the Venn diagram below show that for $p \vee q$ to be true, either p or q must be true. $p \vee q$ is only false if **both** p and q are false.

p	r	$p \vee r$
T	T	T
T	F	T
F	T	T
F	F	F



Again, it might help you to recognise that a disjunction in logic is similar to a union in set theory. In set theory, $A \cup B$ means 'the union of A and B' or 'A or B'. In logic, $p \vee q$ means 'p or q'.

Exclusive disjunction

When two propositions are connected by the phrase 'or, but not both', the compound statement is called an **exclusive disjunction**. The logic symbol for 'or, but not both' is '$\underline{\vee}$'.

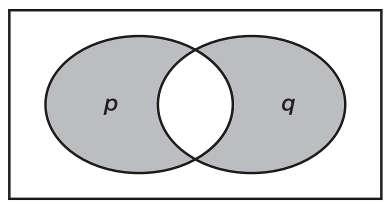
For the propositions p: It is raining

r: It is sunny

the compound statement 'It is raining or it is sunny, but not both' becomes $p \underline{\vee} r$.

The truth table and Venn diagram below show that for $p \underline{\vee} r$ to be true, **either** p is true **or** r is true, but p and r should **not both** be true.

p	r	$p \underline{\vee} r$
T	T	F
T	F	T
F	T	T
F	F	F



Summary

Compound statement	Symbol	What it means	When it is true
Negation	$\neg p$	not p	True when p is false.
Conjunction	$p \wedge q$	p and q	True when both p and q are true.
Disjunction	$p \vee q$	p or q	True when either p or q, or both p and q, are true.
Exclusive disjunction	$p \underline{\vee} q$	p or q, but not both	True when either p or q is true, but not both.

So far we have explained compound statements where each of the propositions consists of words. However, compound statements can be created from propositions that use numbers as well as from those that use words. You should be able to work with both kinds. You also need to be able to interpret a proposition from symbolic notation into words, and from words into symbolic notation.

Worked example 9.2

Q. p: x is a square number

q: $x > 4$

(a) Express these compound statements in symbols:

(i) x is not a square number

(ii) x is a square number and x is greater than 4

(iii) x is a square number or x is greater than 4, but not both

(b) Express these compound statements in words:

(i) $p \vee q$ (ii) $\neg p \wedge \neg q$ (iii) $\neg p \veebar q$

A. (a) (i) $\neg p$

'x is not a square number' is the opposite of the proposition p, so it is the negation of p.

(ii) $p \wedge q$

The connective in the compound statement is 'and', which means this is a conjunction.

(iii) $p \veebar q$

The connective in the compound statements is 'or, but not both', which means this is an exclusive disjunction.

(b) (i) x is a square number or x is greater than 4.

$\vee$ is the symbolic notation for disjunction, which means 'or'.

(ii) x is not a square number and x is not greater than 4.

$\neg p$ indicates that it is the negation of p, the opposite, which is that x is not a square number; $\wedge$ indicates a conjunction 'and'; and $\neg q$ indicates that it is the negation of q, which is x is not greater than 4. Put it all together.

Be careful – the opposite of $x > 4$ is **not** simply $x < 4$; if x is not greater than 4 it could be equal to 4, or less than 4. The true opposite of $x > 4$ is 'x is not greater than 4'.

(iii) x is not a square number or x is greater than 4, but not both.

$\neg p$ indicates that x is not a square number; $\veebar$ is the symbolic notation for exclusive disjunction, which means 'or, but not both'; q is $x > 4$. Put it all together; the 'but not both' comes at the end of the statement whilst the connective 'or' goes between the two propositions.

Exercise 9.2

1. The propositions p, q, and r are defined as follows:

p: Ken plays tennis.

q: The sun is shining.

r: It is hot.

Write these sentences in symbolic notation:

(a) Ken does not play tennis.

(b) It is hot, or Ken plays tennis, but not both.

(c) The sun is shining and it is hot.

(d) It is not hot and the sun is shining.

2. The propositions p, q and r are defined as follows:

p: x is a prime number.

q: $x < 100$.

r: x is a 2-digit number.

Write in words the following statements:

(a) $\neg p$ (b) $p \wedge q$ (c) $\neg p \vee r$

(d) $\neg p \vee \neg r$ (e) $\neg(p \wedge q)$

3. Three logic propositions are given below.

p: Jenny hates football.

q: Jenny watches Sky Sports.

r: Jenny watches the Comedy Channel.

Write the following symbolic statements in words:

(a) $p \wedge \neg q$ (b) $\neg p \wedge q$ (c) $q \veebar r$

4. Propositions p, q and r are defined as follows:

p: Simon is good at Mathematics.

q: Simon does homework regularly.

r: Simon has passed his Mathematics test.

Write symbolic statements for the following sentences:

(a) Simon does his homework regularly and he is good at Mathematics.

(b) Simon does not do homework regularly and has failed his Mathematics test.

(c) Either Simon is not good at Mathematics or he does not do homework regularly.

(d) Simon is not good at Mathematics and he has failed his Mathematics test.

9.3 Implication and equivalence

Implication and equivalence create compound statements where one proposition leads into another.

Implication

When two propositions are connected by the words 'If…, then…' the compound statement is called an **implication**. The logic symbol for 'if…, then…' is '$\Rightarrow$'.

For example, consider the propositions:

p: You play my favourite music.

q: I buy you a soda.

The compound statement, '**If** you play my favourite music **then** I buy you a soda', becomes: $p \Rightarrow q$

p is called the antecedent and q is called the consequent.

exam tip

It is important to use the words 'If…, then…' and not 'therefore'. 'It is raining therefore I have my umbrella' is not the same as 'If it is raining, then I have my umbrella'. 'Therefore' suggests a different relationship between the rain and the umbrella.

The truth table shows that $p \Rightarrow q$ is always true unless p is true and q is false.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In general, an implication is only false if the first proposition is true and the second proposition is false.

If the first implication is false, then the truth of the second is not relevant; it can be true or false, and the overall implication statement is true.

Another way to understand the argument is to think it through as shown below; an argument that involves a promise will help you to remember.

p	q	What does this mean in words?	What is the truth of the implication $p \Rightarrow q$?
T	T	If you play my favourite music, then I buy you a soda.	True. You play my favourite music and I buy you a soda.
T	F	If you play my favourite music, then I do not buy you a soda.	False. You play my favourite music but I do not buy you a soda. According to the implication statement, q cannot be false if p is true, so the overall statement is false.
F	T	If you do not play my favourite music, then I buy you a soda.	True. You do not play my favourite music but I can still buy you a soda. The second statement is not relevant because the first is false; q could still be true even if p is false so the original implication statement is still true.
F	T	If you do not play my favourite music, then I do not buy you a soda.	True. You do not play my favourite music so I do not buy you a soda. Again the truth of the second statement is not relevant because the first statement is false; q could be true or false if p is false, so the implication statement '$p \Rightarrow q$' is still true.

Equivalence

When two propositions are connected by the words 'If and only if…' the compound statement is called an **equivalence**. The logic symbol for 'if and only if' is the double headed arrow '$\Leftrightarrow$'.

When two propositions p and q are such that $p \Rightarrow q$ (if p then q) and $q \Rightarrow p$ (if q then p), the propositions are said to be equivalent.

For example, consider the propositions:

p: Ella passes her exams.

q: Her mother cooks her favourite meal.

$p \Rightarrow q$ says that if Ella passes her exams then her mother cooks her favourite meal.

$q \Rightarrow p$ says that if her mother cooks her favourite meal then Ella passes her exams.

The two statements can be put together, and the compound statement becomes:

$p \Leftrightarrow q$: 'If and only if Ella passes her exams her mother cooks her favourite meal.'

$q \Leftrightarrow p$: 'If and only if her mother cooks her favourite meal, Ella passes her exams'.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Equivalence is only true if the first proposition implies the second and the second implies the first. In other words, an equivalence is only true if both p and q are true or both p and q are false.

Worked example 9.3

Q. Consider the following logic statements:

p: x is a square number.

q: $x > 0$.

Write the following logic statements in words:

(a) $p \Rightarrow \neg q$ (b) $\neg p \Rightarrow \neg q$ (c) $\neg p \Leftrightarrow q$.

A. (a) If x is a square number, then x is not greater than zero.

(b) If x is a not a square number, then x is not greater than zero.

(c) If and only if x is a not a square number, x is greater than zero.

Summary

Compound statement	Symbolic notation	What it means	When it is true
Implication	$p \Rightarrow q$	'If…, then…'	True unless p is true and q is false.
Equivalence	$p \Leftrightarrow q$	'If and only if…'	True when both p and q are true or both p and q are false.

Exercise 9.3

1. Three logic propositions are defined as follows:

p: Veejay revises for his test.

q: Veejay attends football training.

r: Veejay passes his test.

Write in words the following symbolic statements:

(a) $q \veebar r$ (b) $p \wedge \neg q$ (c) $\neg p \wedge q$

(d) $p \Rightarrow \neg q$ (e) $p \Rightarrow r$ (f) $\neg p \Rightarrow \neg r$.

2. Consider the following logic statements:

   p: The weather is cloudy.

   q: I will ride my bike into town.

   r: I will take my umbrella.

   Write the following in symbolic notation:

   (a) If the weather is cloudy, then I will take my umbrella.

   (b) If I ride my bike into town, then I will not take my umbrella.

   (c) If I do not ride my bike into town, then it is cloudy.

3. Suppose p represents 'a triangle is isosceles' and q represents 'a triangle has two equal sides.' Write the following statements using symbolic notation:

   (a) If a triangle is isosceles then it has two equal sides.

   (b) If a triangle does not have two equal sides, then it is not isosceles.

   (c) A triangle is isosceles if and only if it has two equal sides.

4. Let p represent: x is a quadrilateral.

   Let q represent: x is 2-D shape which has a pair of parallel sides.

   Let r represent: x is a parallelogram.

   Write in words the following symbolic statements and indicate whether they are true or false:

   (a) $(p \wedge q) \Rightarrow r$ (b) $r \Rightarrow p$

   (c) $r \Rightarrow q$ (d) $(p \wedge q) \Leftrightarrow r$

9.4 Using truth tables

As you have seen, a truth table helps you to consider every possible outcome for any logical statement. The short truth tables that we have used as diagrams to explain and illustrate the connectives between two propositions can be expanded to illustrate a logical argument and to test its truth.

exam tip

The combined table is in your Formula booklet. Make sure that you have understood each short table before you start building the larger tables and using them.

$a = \pi r^2$

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

The tables look complicated, but if you work carefully and use the basic rules column by column, you should be able to build them with confidence.

Tables for two propositions

If there are two propositions then there are only four possible combinations of the two propositions, and the first two columns of the truth table are always the same:

- both propositions are true: p is true, and q is true (1)
- both propositions are false: p is false, and q is false (2)
- one proposition is true and the other false: p is true, and q is false (3)
  p is false, and q is true. (4)

p	q
T	T
T	F
F	T
F	F

If you write this in the columns as shown above, it makes the rest of a larger table easier to construct. Extra columns can be added for each part of the argument, with the last column used for the final summary.

The following examples show you how to build up the truth tables that you have already met for two propositions, and how to use them to test logical statements. For example, is the statement 'if I revise for my test, then I will go and play football' the same as the statement 'if I go and play football, then I will revise for my test'? In logic terms, is $p \Rightarrow q$ the same statement as $q \Rightarrow p$?

FF You will learn more about logical equivalence in section 9.5.

Here is a reminder of the connectives you have met so far:

Compound statement	Symbolic notation	What it means	When it is true
Negation	$\neg p$	not p	True when p is false.
Conjunction	$p \wedge q$	p and q	True when both p and q are true.
Disjunction	$p \vee q$	p or q	True when either p or q or both p and q are true.
Exclusive disjunction	$p \veebar q$	p or q, but not both	True when either p or q is true, but not both.
Implication	$p \Rightarrow q$	'If…, then…'	True unless p is true and q is false.
Equivalence	$p \Leftrightarrow q$	'If and only if…'	True when both p and q are true or both p and q are false.

Testing if $p \Rightarrow q$ is logically equivalent to $q \Rightarrow p$

You may think that $p \Rightarrow q$ is exactly the same as, or logically equivalent to, $q \Rightarrow p$. You can check using truth tables. Create a truth table for $p \Rightarrow q$; this will have the columns p, q, and $p \Rightarrow q$. Create a truth table for $q \Rightarrow p$; this will have the columns q, p, and $q \Rightarrow p$.

Start both tables with the four possible combinations of p and q shown on page 269, and then work out the truth of the statement in the third column. We have already worked through the table for $p \Rightarrow q$ in the section on implication statements. Let's work through the truth table for $q \Rightarrow p$:

Fill in the first two columns as per the general table for two propositions on page 269.

p	q	$q \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

If q is true and p is true, then $q \Rightarrow p$ is true.

If q is false, then the truth of p is not relevant and $q \Rightarrow p$ is true. This also applies to the last row.

If q is true then p cannot be false; so $q \Rightarrow p$ is false.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$q \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Look at the final columns in the two tables. They are not the same, so $p \Rightarrow q$ and $q \Rightarrow p$ are not logically equivalent.

hint

For $q \Rightarrow p$ consider the first term to be the second term, i.e. in this case q acts like a p and q acts like a p.

Testing if $\neg(p \wedge q)$ is logically equivalent to $\neg p \wedge \neg q$

Logical statements can look as though they should follow the usual rules of algebra, especially if they contain brackets. But this is not necessarily the case.

Take the following statements:

is $\neg(p \wedge q)$ logically equivalent to $\neg p \wedge \neg q$?

If we followed the usual rules of algebra we would expect these statements to be equivalent as they both 'read' as 'not p and not q'. Let's use truth tables to check if they are **logically** equivalent.

Set up two truth tables, one for $\neg(p \wedge q)$ and one for $\neg p \wedge \neg q$. Start with the set columns for p and then q in both tables. For the $\neg(p \wedge q)$ table add the column '$p \wedge q$'. For the $\neg p \wedge \neg q$ table, $\neg p$ and $\neg q$ each have their own column. End each table with the overall statement.

$p \wedge q$ is only true when both p and q are true; $\neg(p \wedge q)$ is only true when $p \wedge q$ is false.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$\neg p \wedge \neg q$ is only true when p and q are false.

This is an example of DeMorgan's Laws, rules that are important in computer programming and digital circuits. The Laws are **not** part of your syllabus.

Look at the final columns. They are not the same, so the two statements are not logically equivalent.

Testing if $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$

We can use truth tables to show that $\neg(p \wedge q)$ is exactly the same as, or logically equivalent to, $\neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Look at the final columns. They are the same, so the two statements are logically equivalent:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Exercise 9.4

1. Two propositions p and q are defined as follows:

p: Donald passed his driving test.

q: Debbie passed her driving test.

(a) Write in symbolic form:

(i) Both Donald and Debbie passed their driving tests.

(ii) Both Donald and Debbie did not pass their driving tests.

(b) Write the following statement in words: $\neg p \vee q$.

(c) Copy and complete the following truth table for the logic statement $\neg p \vee q$.

p	q	$\neg p$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

2. Consider the statements:

p: Arsenal defends well.

q: Arsenal will win the match.

(a) Write the following propositions using symbolic notation:

(i) If Arsenal does not defend well they will not win the match.

(ii) Arsenal will win the match if and only if they defend well.

(b) Copy and complete the truth table, using the information from the statement made in part (a) (i).

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$
T	T			
T	F			
F	T			
F	F			

Tables for three propositions

The basic table is:

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

If you are using three propositions, then the possible combinations of the propositions increases from four to eight. The truth table needs to be set up so that all eight possible combinations are included. Recall that the usual letters to denote propositions are p, q, r, s; here we have used r to denote the third proposition.

If you always start with the same three columns and build the argument stage by stage, the tables are straightforward. Look at the pattern of T and F in the columns.

This table can now be used to construct the truth table for a compound statement involving three propositions, and to test whether the argument is true in every situation.

Is $p \wedge \neg q \Rightarrow r$ always true?

Add an extra column for $\neg q$, then one for $p \wedge \neg q$, and a final column for the whole statement. Work down each column row by row.

p	q	r	$\neg q$	$p \wedge \neg q$	$p \wedge \neg q \Rightarrow r$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Remember that if the truth of the first statement is false then the truth of the second statement is irrelevant and the implication statement is true.

'$p \wedge \neg q \Rightarrow r$' cannot be true if '$r$' is false, so the final statement is false.

The final column consists mainly of 'true' but there is also a 'false', so this argument is not true in **all** situations.

You can also use the truth tables to construct arguments when you are given the propositions in full.

'It is cold and the ice is thick' is written $(p \wedge q)$.

'If and only if it is cold and the ice is thick, then I will go skating' is written $(p \wedge q) \Leftrightarrow r$.

Set up the first three columns as per the basic truth table for three propositions on page 272.

Worked example 9.4

Q. p: It is cold.

q: The ice is thick.

r: I will go skating.

Use a truth table to test the statement, 'if and only if it is cold and the ice is thick, then I will go skating'.

A.

p	q	r	$p \wedge q$	$p \wedge q \Leftrightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	F	F
F	F	F	F	T

$(p \wedge q) \Leftrightarrow r$ is not a valid argument. The final column of the truth table contains both true and false conclusions. If an argument is valid the entries in the final column must all be true.

Exercise 9.5

exam tip

Questions using three propositions may be set with some of the columns partly filled in. You can use the short truth tables in your Formula booklet to help you complete them.

1. For an IB Revision Course students have three choices:

e: Economics

m: Mathematics

s: a Science subject.

Students choose two subjects which **must** include Mathematics and either Economics or a Science subject, but not both.

(a) Write the sentence above, using symbolic notation.

(b) Write in words $\neg s \Rightarrow e$.

(c) Complete the truth table.

e	s	$\neg e$	$\neg e \Rightarrow s$
T	T		
T	F		
F	T		
F	F		

2. Consider the statements:

p: It is the weekend.

q: I will go to the cinema.

(a) Write down, in words, the meaning of $q \Rightarrow \neg p$.

(b) Complete the truth table.

p	q	$\neg p$	$q \Rightarrow \neg p$
T	T		
T	F		
F	T		
F	F		

3. Three propositions are defined as:

p: The exam paper was easy.

q: The grade boundaries were low.

r: Joe performed well in the exam.

Samantha says, 'If Joe performed well in the exam then either the exam paper was easy or the grade boundaries were low.'

Complete the truth table for Samantha's statement and comment on the logical validity of the statement.

p	q	r	$p \veebar q$	$r \Rightarrow (p \veebar q)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

4. Let p represent: Boris is a rugby player, let q represent: Boris has the rugby ball and let r represent: Boris has a football.

(a) Write the following using symbolic notation:

(i) If Boris is a rugby player then he has the rugby ball.

(ii) Boris has the rugby ball and he has not got a football.

(iii) If Boris is not a rugby player then he has a football.

(b) Write the following argument in words:

$$\neg r \Rightarrow (p \vee q)$$

(c) Construct a truth table for the argument in part (b) using the values below for p, q, r and $\neg r$.

p	q	r	$\neg r$	$p \vee q$	$\neg r \Rightarrow (p \vee q)$
T	T	T	F		
T	T	F	T		
T	F	T	F		
T	F	F	T		
F	T	T	F		
F	T	F	T		
F	F	T	F		
F	F	F	T		

9.5 Logical equivalence, tautology and contradiction

Logical equivalence, **tautology** and **contradiction** are all tested using truth tables. In each case it is the final column of the table that is the most important as it will give you the result.

If you are testing two compound statements for **logical equivalence**, you are looking to see whether they are interchangeable. The statements may be in words or symbols but if one is true, so is the other. If one is false, then the other statement is false too.

A logical **tautology** is always true and is never shown to be false, even if the initial propositions are false.

A logical **contradiction** is never true and cannot be shown to be true.

Logical equivalence

You have already seen some examples of testing the logical equivalence of two statements by using two truth tables, one for each statement. It is best to concentrate on completing each table separately; if they are logically equivalent then the final column of both truth tables will be exactly the same. Some of the results may surprise you but if you think about a likely result and check your tables carefully, you will see that it is correct.

Not all the logic questions set in assignments or examinations use words; they may also be set as propositions that are not defined. Problems like this can be easier to work through than questions using words, as you do not have to think about whether the statements you are testing are true in a practical way and whether or not you agree with them.

hint

Do not get confused between the terms 'equivalence' and 'logically equivalent/logical equivalence'.

'Equivalence' is **one compound statement** connected by the symbol $\Leftrightarrow$ or by the words 'if and only if…'.

Logically equivalent is when **two compound statements** are exactly the same.

Worked example 9.5

Q. Is $(p \Rightarrow q)$ logically equivalent to $(\neg p \vee q)$?

A.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$(\neg p \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Create two truth tables. One for $p \Rightarrow q$ and one for $(\neg p \vee q)$. Compare the final columns of the two tables.

The final columns are exactly the same, so the statements are logically equivalent:

$$(p \Rightarrow q) \equiv (\neg p \vee q)$$

Logical tautology

If a compound statement is always true, it is called a **tautology**. In this case all the values in the final column will be true. Some of the results may seem to be obvious but are worth looking at to help your understanding; others do not seem so obvious.

The first example is set using words, but the second is an illustration using the connections established in earlier sections of this chapter.

Worked example 9.6

Q. If q: swans are white

and

$\neg q$: swans are not white

show that $(q \vee \neg q)$ is a tautology.

A.

q	$\neg q$	$q \vee \neg q$
T	F	T
F	T	T

If q: swans are white

$\neg q$: swans are not white

then $(q \vee \neg q)$ tells you that all swans are white or not white – it is a tautology.

This truth table looks very complex. But if you use the techniques you learned for constructing tables with two propositions and then complete it column by column, you will see that it is straightforward. Make sure that you have a separate column for each part of the statement.

Use '$p \wedge q$' and '$(p \wedge q) \Rightarrow r$' for the first part of the argument; use '$q \Rightarrow r$' and '$p \Rightarrow (q \Rightarrow r)$' for the second part and '$\{(p \wedge q) \Rightarrow r\} \Leftrightarrow \{p \Rightarrow (q \Rightarrow r)\}$' for the complete statement.

Worked example 9.7

Q. Show that $\{(p \wedge q) \Rightarrow r\} \Leftrightarrow \{p \Rightarrow (q \Rightarrow r)\}$ is a tautology.

A.

p	q	r	$p \wedge q$	$(p \wedge q) \Rightarrow r$	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$\{(p \wedge q) \Rightarrow r\} \Leftrightarrow \{p \Rightarrow (q \Rightarrow r)\}$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Logical contradiction

This is the reverse of tautology. If a compound statement is always false, it is called a **contradiction**. When you complete the truth table all the results in the final column are false. As before, some of the results might appear to be obvious while others might be surprising.

Worked example 9.8

Q. If q: swans are white

$\neg q$: swans are not white

(a) show that $(q \wedge \neg q)$ is a contradiction.

A. (a)

q	$\neg q$	$q \wedge \neg q$
T	F	F
F	T	F

If q: swans are white

$\neg q$: swans are not white

then $(q \wedge \neg q)$ tells you that all swans are white and not white – it is a contradiction.

(b) Use a truth table to show that $(p \vee q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

A. (b)

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	$(p \vee q) \wedge (\neg p \wedge \neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

The final column of the truth table contains only false conclusions, therefore the statement $(p \vee q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

The difference between inductive and deductive logic is studied in the Theory of Knowledge course. The notion of fallacies is also important.

The study of logic helps when you are preparing presentations and essays. Theory of Knowledge essays are 1600 words long, so a logical structure will help you to organise the ideas you need together.

Exercise 9.6

1. Consider the statements:

p: Oliver likes horse riding.

q: Ulrika likes skiing.

For each of the symbolic statements, (a) to (e):

(i) construct the corresponding truth table

(ii) indicate whether the statement is a tautology, a contradiction, or neither.

(a) $\neg(p \wedge \neg q)$

(b) $p \vee \neg(p \wedge q)$

(c) $(p \Rightarrow q) \wedge (\neg p \wedge q)$

(d) $\neg(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$

(e) $p \wedge (\neg q \wedge p)$

2. Three propositions are defined as follows:

p: The bad weather continues.

q: This week's cricket match will be cancelled.

r: I will watch football.

(a) Write in words the symbolic statements:

(i) $\neg p \vee q$

(ii) $\neg p \wedge \neg q$

(iii) $\neg q \Leftrightarrow \neg p$

(iv) $(p \wedge q) \veebar (q \wedge \neg p)$.

(b) Construct truth tables for each pair of statements and indicate whether or not they are logically equivalent.

(i) $p \Rightarrow q$ and $\neg p \vee q$

(ii) $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$

(iii) $p \Leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

(iv) $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$

3. The two propositions p and q are defined as follows:

p: The internet is not working.

q: I check my emails.

(a) Write in words the following statements:

(i) $p \Rightarrow \neg q$

(ii) $\neg p \Rightarrow q$.

(b) Construct a truth table for the compound proposition given below.
$(p \Rightarrow \neg q) \vee (\neg p \Rightarrow q)$

(c) Using your table from part (b) state whether the statement is a contradiction, a tautology, or neither.

9.6 Converse, inverse and contrapositive

The implication $p \Rightarrow q$ has three closely related statements: the **converse**, the **inverse** and the **contrapositive**.

If $p \Rightarrow q$, then $q \Rightarrow p$ is the converse
$\neg p \Rightarrow \neg q$ is the inverse
$\neg q \Rightarrow \neg p$ is the contrapositive.

Converse

To form the **converse** of the statement $p \Rightarrow q$, you **convert** p to q and q to p.

Take the propositions:

p: Poppa eats spinach.

q: Poppa is strong.

$p \Rightarrow q$ gives the statement, 'If Poppa eats spinach then Poppa is strong'.

$q \Rightarrow p$ gives the statement, 'If Poppa is strong then Poppa eats spinach'.

If $p \Rightarrow q$ is true, then $q \Rightarrow p$ is only true when both p and q are true, both p and q are false, or p is true and q is false. It is not true in every case.

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

hint

You might find it helpful to remind yourself of the truth table for implication in section 9.3.

Inverse

To form the **inverse** of the statement $p \Rightarrow q$, you **invert** p to its negative $\neg p$ and invert q to its negative $\neg q$.

Using the propositions:

p: Poppa eats spinach.

q: Poppa is strong.

$p \Rightarrow q$ gives the statement, 'If Poppa eats spinach then Poppa is strong'.

$\neg p \Rightarrow \neg q$ gives the statement, 'If Poppa does not eat spinach then Poppa is not strong'.

If $p \Rightarrow q$ is true, then $\neg p \Rightarrow \neg q$ is only true when both p and q are true, both p and q are false, or p is true and q is false. It is not true in every case.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Contrapositive

To form the **contrapositive** of the original statement, you combine both the converse and the inverse. The propositions are turned round, and then negated.

$p \Rightarrow q$ becomes $q \Rightarrow p$, then $\neg q \Rightarrow \neg p$

For example, consider the propositions:

p: Poppa eats spinach.

q: Poppa is strong.

$p \Rightarrow q$ gives the statement, 'If Poppa eats spinach then Poppa is strong'.

$\neg q \Rightarrow \neg p$ gives the statement, 'If Poppa is not strong then Poppa does not eat spinach'.

The result of combining the inverse and converse is that the contrapositive is now logically equivalent to the original statement.

'If Poppa eats spinach then Poppa is strong', is logically equivalent to, 'If Poppa is not strong then Poppa does not eat spinach', even if this seems unlikely given the propositions.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The logical equivalence between the converse, inverse and contrapositive can be seen clearly in their truth tables.

Truth tables for the converse and inverse

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$q \Rightarrow p$	$\neg p \Rightarrow \neg q$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Truth tables for the contrapositive and the original implication

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logic is used in digital circuits and design technology as well as in computer programming.

Worked example 9.9

Q. Consider the propositions p: x is an even number.

q: x can be divided by 2.

Give the converse, inverse and contrapositive statements. What do you notice?

A. Original: $p \Rightarrow q$

If x is an even number, then x can be divided by 2.

Converse: $q \Rightarrow p$

If x can be divided by 2, then x is an even number.

Inverse: $\neg p \Rightarrow \neg q$

If x is not an even number, then x cannot be divided by 2.

Contrapositive: $\neg q \Rightarrow \neg p$

If x cannot be divided by 2, then x is not an even number.

For these specific propositions, the original statement, the converse, the inverse and the contrapositive are all true according to the rules of mathematical logic **and** the rules of arithmetic, for answers that are integers.

Worked example 9.10

Q. Let p and q represent the propositions

p: Henry practises his flute.

q: Henry plays in the band.

Write the following statements in symbols. Which statement is the converse? Which statement is the inverse?

(a) If Henry plays in the band, then Henry practises his flute.

(b) If Henry does not practise his flute, then Henry does not play in the band.

(c) Write the contrapositive of these propositions in words.

A. (a) $q \Rightarrow p$ This is the converse, as p and q have been reversed.

(b) $\neg p \Rightarrow \neg q$ This is the inverse, as both p and q have been negated.

(c) If Henry does not play in the band, then Henry does not practise his flute.

Exercise 9.7

1. Let p, q and r represent the propositions:

   p: The music is good.

   q: I feel like dancing.

   r: I dance to the music.

   Write the following in symbols and then in words:

   (a) the inverse of $p \Rightarrow r$

   (b) the converse of $p \Rightarrow r$

   (c) the contrapositive of $q \Rightarrow r$.

2. For each of the statements (a) to (e) write in words the corresponding:

   (i) inverse (ii) converse (iii) contrapositive.

   (a) If you listen attentively in class, then you perform well in tests.

   (b) If you like current affairs, then you listen to news regularly.

   (c) If you are taught by Mrs Brown, then you are brilliant at Logic.

   (d) If Sandra is unwell, then she cannot play in the netball match.

   (e) If Andrew is good at languages, then he can be a tourist guide.

3. Suppose p represents 'Grandma goes to the dentist' and q represents 'Grandma visits Aunt Sally'.

   (a) Write in words the converse of $p \Rightarrow q$.

   (b) Write the following proposition in symbolic form.

   'If Grandma does not visit Aunt Sally, then she goes to the dentist'.

   (c) Is the proposition in part (b) the inverse, converse or the contrapositive of the proposition in part (a)?

4. Jasmine makes the statement 'If a shape is a rectangle, then it is a parallelogram.'

   (a) For this statement, write in words its:

   (i) converse

   (ii) inverse

   (iii) contrapositive.

   (b) Which of the statements in part (a) is true?

Summary

You should know:

- the basics of symbolic logic
- the definition of propositions and the symbolic notation used to describe them
- what compound statements are and the different types of connectives between them
- how to illustrate compound statements using truth tables and Venn diagrams
- how to use truth tables to test arguments
- how to use truth tables to demonstrate the concepts of logical equivalence, contradiction and tautology
- how to construct converse, inverse and contrapositive statements.

Mixed examination practice

Exam-style questions

1. Consider the following logic statements:

p: My laptop is broken.

q: My laptop is fixed.

r: I will finish writing up my Portfolio task.

(a) Write in words the following symbolic statements;

(i) $\neg q \wedge \neg r$

(ii) $q => r$

(iii) $r \Leftrightarrow q$.

(b) Write the following statements, using symbolic notation:

(i) My laptop is broken and it is not fixed.

(ii) My laptop is broken and I will not finish writing up my Portfolio task.

2. Consider the following statements:

p: New Year is approaching.

q: I will shop for presents.

(a) Write down, in words, the meaning of $p \Rightarrow q$.

(b) Copy and complete the truth table.

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$
T	T			
T	F			
F	T			
F	F			

3. Consider the following logic statements:

p: I do not save enough money.

q: I buy a new car.

(a) Write the expression $\neg p \Rightarrow q$ as a logic statement.

(b) Write the following statement in logic symbols:

'I save enough money and I do not buy a new car.'

(c) Copy and complete the truth table.

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow q$	$\neg p \wedge \neg q$
T	T	F	F	F	F
T	F	F	T		
F	T	T	F		
F	F	T	T		

4. For each of the statements, write in words the corresponding:

(i) inverse (ii) converse (iii) contrapositive.

(a) If Elliot passes his driving test then his dad will buy him a new car.

(b) If it snows heavily tonight then the roads will not be busy tomorrow morning.

(c) If the recession continues then unemployment will remain high.

5. Consider the two propositions

p: Ali goes to the Homework Club.

q: Ali goes home early.

Nadia says: 'If Ali goes to the Homework Club, then Ali does not go home early'.

(a) Write Nadia's statement in symbolic form.

(b) Write, in symbolic form, the contrapositive of Nadia's statement.

6. Three logic propositions are given below:

p: x is a polygon.

q: x has equal sides and equal angles.

r: x is a regular polygon.

Write in words the following symbolic statements and indicate whether they are true or false:

(a) $q => r$ (b) $r \Leftrightarrow (p \wedge q)$ (c) $p \Leftrightarrow q$.

7. Let p and q be the statements

p: Marco is a member of the debating society.

q: Marco enjoys debating.

(a) Consider the following logic statement:

'If Marco is a member of the debating society then he enjoys debating.'

(i) Write down in words the inverse of the statement.

(ii) Write down in words the converse of the statement.

(b) Construct truth tables for the following statements:

(i) $p \Rightarrow q$ (ii) $\neg p \Rightarrow \neg q$

(iii) $p \vee \neg q$ (iv) $\neg p \wedge q$.

(c) Which of the statements in part (b) are logically equivalent?

Past paper questions

1. Complete the truth table for the compound proposition $(p \wedge \neg q) \Rightarrow (p \vee q)$.

p	q	$\neg q$	$(p \wedge \neg q)$	$(p \vee q)$	$(p \wedge \neg q) \Rightarrow (p \vee q)$
T	T	F	F		
T	F	T	T		
F	T	F		T	
F	F		F	F	

[Total 8 marks]

[Nov 2005, Paper 1, Question 11] (© *IB Organization 2005*)

2. (a) **Copy** and **complete** the table below by filling in the three empty columns.

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$(p \vee q) \wedge \neg p \Rightarrow q$
T	T	T	T			
T	F	F	T			
F	T	F	T			
F	F	F	F			

[3 marks]

(b) What word is used to describe the argument $(p \vee q) \wedge \neg p \Rightarrow q$? *[1 mark]*

[Total 4 marks]

[May 2005, Paper 2, Question 3(ii)] (© *IB Organization 2005*)

3. The truth table below shows the truth-values for the proposition

$p \veebar q \Rightarrow \neg p \veebar \neg q$.

p	q	$\neg p$	$\neg q$	$p \veebar q$	$\neg p \veebar \neg q$	$p \veebar q \Rightarrow \neg p \veebar \neg q$
T	T	F	F		F	
T	F	F		T	T	T
F	T	T	F	T	T	T
F	F	T	T	F		T

(a) Explain the distinction between the compound propositions, $p \veebar q$ and $\neg p \veebar \neg q$.

(b) Fill in the four missing truth-values on the table.

(c) State whether the proposition $p \veebar q \Rightarrow \neg p \veebar \neg q$ is a tautology, a contradiction or neither.

[Total 6 marks]

[May 2007, Paper 1, Question 4] (© *IB Organization 2007*)

4. (a) (i) Complete the truth table below.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T			F	F	
T	F			F	T	
F	T			T	F	
F	F			T	T	

(ii) State whether the compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equivalent. *[4 marks]*

Consider the following propositions.

p: *Amy eats sweets*

q: *Amy goes swimming.*

(b) Write, in symbolic form, the following proposition.

Amy either eats sweets or goes swimming, but not both. *[2 marks]*

[Total 6 marks]

[May 2008, Paper 1, TZ2, Question 1] (© *IB Organization 2008*)

Chapter 10 Probability

In this chapter you will learn:

- what the sample space is and how to construct sample space diagrams
- how to calculate expected values
- how to calculate probabilities of events and their complements
- how to find the probabilities of combined, mutually exclusive and independent events
- how to use tree diagrams, Venn diagrams, sample space diagrams and tables of outcomes to calculate probabilities
- that probability 'with replacement' is not the same as probability 'without replacement'
- how to calculate conditional probability from sample space diagrams.

Liber de Ludo Aleae was written by the Italian physician and mathematician Girolamo Cardano (1501–1576), but it was not published until after his death. Do you think it is significant that probability can be determined by mathematics? Are there still occasions when people believe that they can determine their future by chance?

Two thousand years ago, Roman soldiers would settle disputes by tossing a coin. They believed that if the coin landed to show a 'head', the result was ordained by the state — or by the gods. Similarly, in India, people would roll dice and respect the decision made by the dice because they believed that was the outcome that the gods desired.

It was not until 1663 that a book was published on the mathematics of chance; it was called *Liber de Ludo Aleae*. The book demonstrated that the outcome of tossing a coin or rolling a dice could be predicted with some accuracy, and probability started to lose its mystery.

10.1 Introduction to probability

Probability is a measure of 'chance'. It gives a numerical representation of the likelihood that a result will be obtained. Probability can be measured as a percentage, a decimal or a fraction. For example, if there is a 1 in 4 chance of something happening, you could represent this in three ways.

25%

$\frac{1}{4}$

0.25

All probabilities lie between zero and one:

- If a result is certain then it has a probability of 1.
- If a result is impossible, it has a probability of 0.

The 'results' of a probability experiment are known as **outcomes**. An individual outcome being investigated is known as an **event**.

Probabilities are normally represented by the letter P with the event enclosed within parentheses. For example, P(red sock) = $\frac{1}{4}$.

Complementary events

If your local weather forecast tells you that there is a 65% chance of a storm tomorrow, what is the chance that there is no storm?

All the probabilities in any situation must add to a total of 1:

P(storm) = 0.65, so P(no storm) = 1 – 0.65 = 0.35

The probability that there is no storm is 35%.

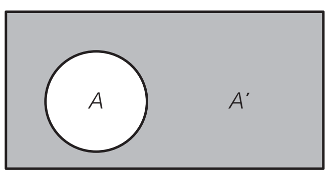
The event 'there is no storm' is the **complement** of the event 'there is a storm'.

In general, $P(A') = 1 - P(A)$

hint

A' is the notation to represent 'not A'

This can be illustrated using a Venn diagram:



The complement of an event consists of all the possibilities that are not in the event.

When you are solving probability problems, it is often useful to look for complementary events, as it can make the calculations easier.

Worked example 10.1

In a normal deck of playing cards, there are four suits: Hearts (red), Diamonds (red), Spades (black) and Clubs (black). Each suit is made up of 13 cards: 2 to 10, Jack, Queen, King and Ace.

Q. Sacha has a deck of cards. The probability of picking an ace is $\frac{4}{52}$. He picks a card at random. What is the probability that he does **not** pick an ace?

Sacha has subtracted the probability of picking an ace from one.

A. $P(\text{Ace}) = \frac{4}{52}$, so $P(A') = 1 - \frac{4}{52} = \frac{48}{52}$

Exercise 10.1

1. The probability of Henry catching his school bus in the mornings is 0.99. What is the probability of Henry missing his bus?

2. Jess has a bag of sweets. The bag contains red, green and brown sweets; the probability of picking a green sweet is $\frac{6}{20}$ and the probability of picking a red sweet is $\frac{9}{20}$. Jess picks a sweet out of the bag randomly (without looking).

   (a) What is the probability of picking a brown sweet?

   (b) Find the probability of **not** picking a brown sweet.

3. There are 24 students in a class. Each student studies only one subject out of History, Geography or Politics. $\frac{5}{12}$ students study History and $\frac{4}{12}$ students study Geography. One student is selected at random to represent the class in the Student Forum.

   (a) What is the probability that the student chosen studies Politics?

   (b) Find the probability that the student chosen does **not** study Politics.

GDC

See '*24.2E Entering fractions*' on page 644 of the GDC chapter for a reminder of how to use your calculator for calculations involving fractions if you need to.

10.2 Sample space diagrams

If one of the Roman soldiers had thrown one coin, what is the probability that it showed a 'head'? As a coin has two sides and the 'head' is only on one side, the probability of throwing a head is 'one out of two' or $\frac{1}{2}$. This is written as P(Head) $= \frac{1}{2}$, where P(Head) means 'the probability of obtaining a head on one throw'. 'Obtaining a head on one throw' is known as an event. 'Obtaining a tail on one throw' is another event. Each of these events is a possible outcome (result) of throwing a coin.

If the Roman soldier had thrown two coins, what is the probability that he would have thrown two heads? 'The probability of obtaining two heads in two throws' is also called an event. It is a possible outcome of throwing two coins. There are other possible outcomes of throwing two coins, for example throwing a head on one coin and a tail on the other coin.

The complete set of possible outcomes from an experiment, such as throwing a die or a coin, is called the **sample space.**

In order to calculate the probability of a given event it is useful to draw a diagram that lists all the possible outcomes. This diagram is called a **sample space diagram**.

Let's return to the Roman soldier who threw two coins; there are only four possible outcomes:

First coin = H; second coin = H

First coin = H; second coin = T

hint

Die or dice? Die is the singular (one), dice is the plural and should be used when you have two or more. This convention is not always kept and 'dice' may be used for one die as well as for several.

First coin = T; second coin = H

First coin = T; second coin = T

The sample space could be represented more neatly using a simple table:

1st throw	H	H	T	T
2nd throw	H	T	H	T

Or written as a set: {(H, H), (H, T), (T, H), (T, T)}

Both sample space diagrams tell us that there are only four possible outcomes, and if you are just as likely to get a H as a T, then the probabilities are as follows:

- P(throwing two heads with one coin) = P(H, H) = $\frac{1}{4}$ (one out of four)
- P(throwing one head and one tail) = P((H, T) or (T, H)) = $\frac{2}{4} = \frac{1}{2}$ (two out of four)
- P(throwing two tails with two coins) = P(T, T) = $\frac{1}{4}$ (one out of four).

Note that in any sample space, all the probabilities will add to a total of one, e.g. $\frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1$.

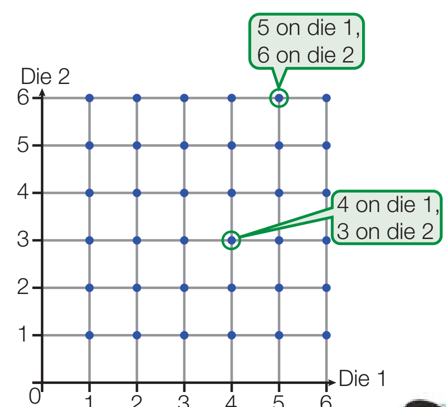
Writing the sample space as a simple list or table is straightforward when the number of possible outcomes is quite small. But when there are lots of possible outcomes, other sample space diagrams might be more appropriate.

Examples of sample space diagrams

As you know, a sample space diagram is a way of listing all the possible outcomes. It can be as simple as a list of the possibilities or it can be much more complicated. The following examples show some of the ways that you can create your own diagram but it is not a comprehensive list. For some questions more than one type of diagram may be used. You can use any of the diagrams but it is important that you can understand any sample space diagram and that you include every possible outcome.

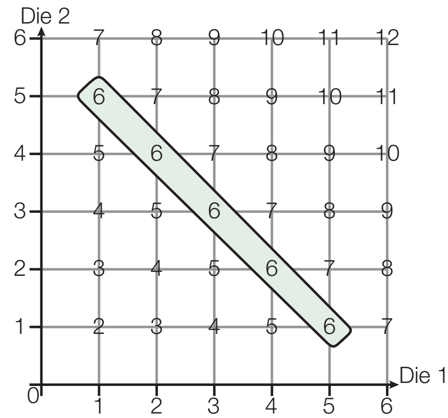
A grid sample space diagram

The grid consists of a vertical and a horizontal axis, like a graph. It is useful for showing the sample space for a **combined event**; a combined event is when two or more single events occur one after the other (for example throwing two coins, two dice, or throwing a coin and a dice).

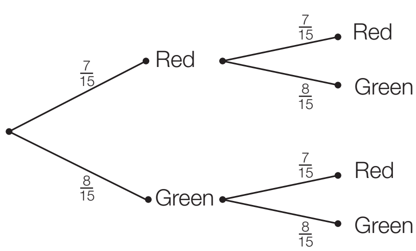


The possible outcomes of each event are listed on the axes and the 'plotted' points show the various combinations of outcomes when two events occur together. It displays all the possible combinations so that you can see them clearly. A diagram like this is useful for problems that involve two dice. You can see in this example that if two dice are thrown, the possible value on each die is plotted along each axis. You can then circle the points you are interested in.

For example, suppose we wanted to know what the probability is of the event 'the sum of the two dice adds up to 6'. We would draw the space sample diagram as shown here and circle all relevant points; in this case there are 5 of them. The total number of points; is 36 (6 along the horizontal axis × 6 along the vertical axis). So the probability of a sum of 6 is 5 out of 36.



A tree diagram



A **tree diagram** shows all the possible successive events in a problem; this is useful for **combined events**, where one event occurs after another. The probability of taking a certain 'branch' of the tree is written along that branch. The probabilities along each branch are multiplied together to get the final probability of a given combined event occurring.

Tables of outcomes

A simple table:

In a simple table, the possible outcomes of a **single event** (throwing only one die or one coin, etc.) are listed along with the probability of obtaining that outcome.

For example, when a fair six-sided die is thrown the possible outcomes are 1, 2, 3, 4, 5 or 6; these are listed as x. As the die is not biased, there is an equal probability of each outcome.

The die has six sides, so $P(1) = P(2) = \ldots = \frac{1}{6}$.

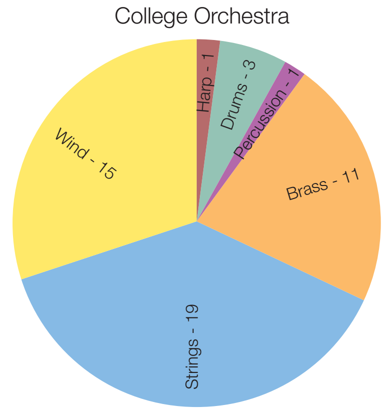
x	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

A two-way table:

Two-way tables display the frequency for two or more sets of data, and allow you to see all the different combinations of that data. They are often called **contingency** tables because within each cell they show the frequency for a given combination of events. These are useful for **combined events**. In this example, we have the number of boys and girls in a particular sports club who enjoy either tennis or athletics. This sample space allows you to calculate the probabilities for various different events.

For example, we can see from the table that 17 out of all 42 (40%) children liked to play tennis but only 5 out of 22 (23%) girls liked to play tennis.

	Tennis	Athletics	Total
Boys	12	8	20
Girls	5	17	22
Total	17	25	42

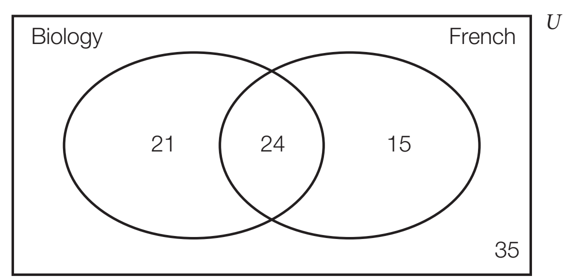


A pie chart

You know from Chapter 5 that pie charts display frequencies as parts of a whole, where the circle represents the total frequency of your data and the sectors represent different measurements. This is useful for **single events**. This chart shows the number of students who play in each section of their college orchestra.

A Venn diagram

A Venn diagram is a good way of displaying information. You know from Chapter 8 that a Venn diagram can be used to illustrate overlap between different sets of data (and you have been told in this chapter that the sample space can be written as a set).



Exercise 10.2

1. Alexandra rolls two four-sided dice, both regular tetrahedra. She adds together the numbers on their bases.

Copy and complete this sample space diagram.

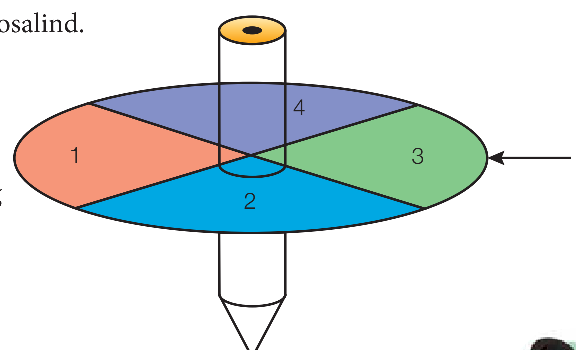
	1	2	3	4
1	2			5
2				
3			6	
4		6		

exam tip

In an examination you can be asked to complete a sample space diagram, or to draw your own. If you draw your own, the examiner will mark any clear diagram and check to ensure that your answer to the problem matches it.

2. For the two vacancies of Head Girl and Deputy Head Girl there are three candidates on the voting list: Bryony, Paige and Rosalind. Complete a sample space diagram to illustrate the possible combinations of the candidates for the two positions.

3. The spinner below has four sectors, marked 1, 2, 3 and 4. Olaf spins the spinner twice and records the number facing the arrow each time. He then adds the two numbers together.



Draw a sample space diagram to illustrate all the possible sums Olaf is likely to get.

4. A triangular spinner has sectors labeled A, B and C. It is spun twice.

(a) What is the total number of possible outcomes?

(b) Complete a full sample space diagram for the possible outcomes.

The spinner is spun three times.

(c) What is the total number of possible outcomes?

10.3 Calculating probability and the expected value

You know that when you draw up a sample space diagram you are listing all the possible outcomes.

When tossing two coins, the event A (tossing two heads) can be written as:

$A = \{(H, H)\}$

There are four different ways that the coins can fall, so all the outcomes can be written as:

$U = \{(H, H), (H, T), (T, H), (T, T)\}$

RR ***U is the universal set; you learned about this in Chapter 8.***

From section 10.2, you know that:

$P(H, H) = \frac{1}{4}$ (one out of four)

$P(H \text{ and } T) = \frac{2}{4}$ (two out of four)

$P(T, T) = \frac{1}{4}$ (one out of four)

This leads us to a general formula for calculating probability:

$a = \pi r^2$

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} \quad \textbf{or} \quad P(A) = \frac{n(A)}{n(U)}$$

In Physics, probability theory is important when studying the position of an electron in an atom. In Biology, it is used in the theory of genetics. For example, Punnett squares are sample space diagrams that show all the possible genetic combinations from the cross of chosen parents. This is illustrated below.

MATERNAL \ PATERNAL	C	c
C	CC	Cc
c	cC	cc

Worked example 10.2

Q. The numbers 1 to 10 are written on separate pieces of card and put into a bag. Sophie puts her hand in the bag and takes out a single card.

(a) What is the probability that she picks a prime number?

(b) What is the probability that she picks a multiple of 5?

(c) What is the probability that she picks a zero?

continued . . .

(d) What is the probability that she picks a number from 1 to 10?

(e) What is the probability that she does not pick a prime number?

A. The sample space diagram in this case is easier to write out using sets:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Prime numbers $= \{2, 3, 5, 7\}$

Multiples of $5 = \{5, 10\}$.

(a) $P(\text{prime number}) = \frac{4}{10}$

There is a total of ten numbers, 4 are prime numbers.

(b) $P(\text{multiple of five}) = \frac{2}{10}$

There are two multiples of five.

(c) $P(\text{zero}) = 0$

The universal set does not contain a zero.

(d) $P(1 \ldots 10) = 1$

A number from 1 to 10 is certain.

(e) $P(\text{no prime number}) = \frac{6}{10}$

If four numbers are prime, six are not (10 – 4); or you can use the fact that 'prime' and 'not prime' are complementary events and calculate $1 - \frac{4}{10}$.

Worked example 10.3

Q. A die with eight faces (an octahedron) has one of a, b, c, d, e, f, g, h written on each face. The die is rolled and the letter on the top face is noted.

(a) Draw a sample space diagram.

(b) Calculate the probability that the die lands with 'f' on the top face.

(c) Calculate the probability that the die lands with a vowel on the top face.

(d) Calculate the probability that the die does not land with a vowel on the top face.

continued . . .

The list of letters in the universal set is used as the space diagram.

A. (a) $U = \{a, b, c, d, e, f, g, h\}$

There is only one 'f'.

(b) $P(f) = \frac{1}{8}$

There are two vowels out of 8 letters.

(c) $P(\text{vowel}) = \frac{2}{8} = \frac{1}{4}$

If there are two vowels, there must be 8 – 2 = 6 consonants. (Note that you could also have subtracted the probability of getting a vowel from 1 to get the probability of 'not a vowel': $1 - \frac{1}{4} = \frac{3}{4}$.)

(d) $P(\text{consonant}) = \frac{6}{8} = \frac{3}{4}$

Worked example 10.4

Q. Two dice are thrown and the scores are added together. Calculate the following probabilities:

(a) P(total of six)

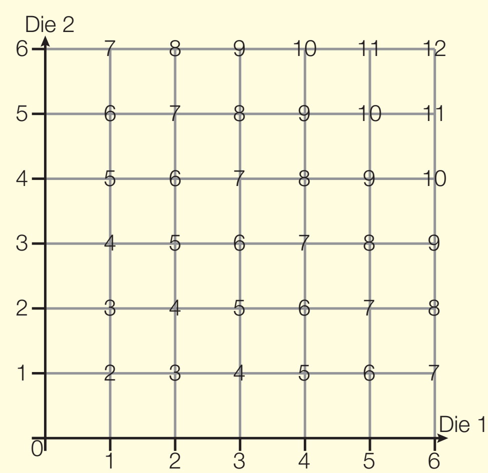
(b) P(total of at least eight)

(c) P(the two scores are the same)

(d) P(total of 1).

A.

First you need to draw a sample space diagram. The best diagram for this problem is a sample space diagram where you can see all the possible combinations and outline the ones that you need in order to help you answer the questions.



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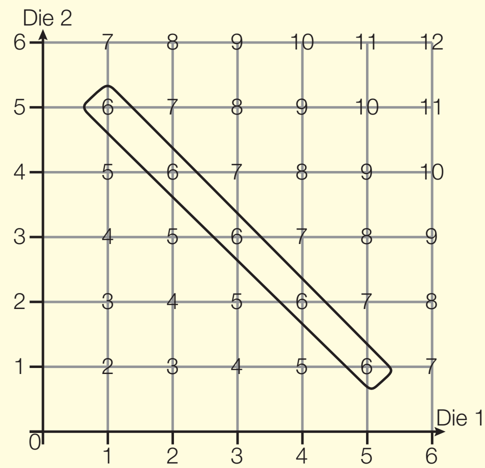
Circle all occurrences of '6'. There are five.

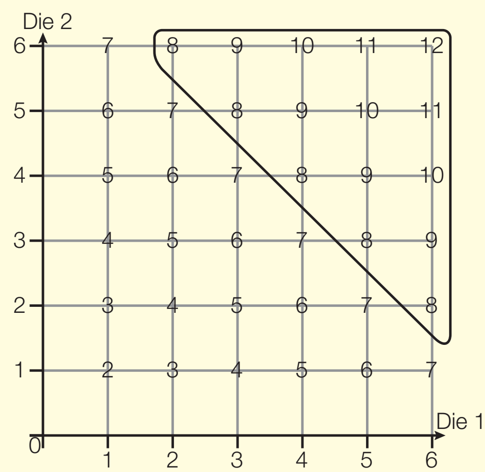
(a)



P(total of six) $= \dfrac{5}{36}$

'At least eight' means 'eight or more', so circle all occurrences of 8, 9, 10, 11 and 12. There are 15.

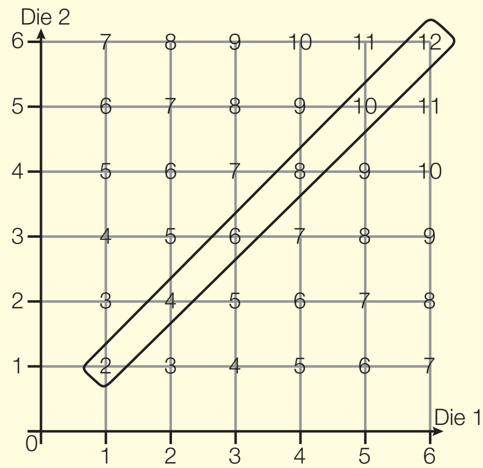
(b)



P(total of at least eight) $= \dfrac{15}{36} = \dfrac{5}{12}$

continued . . .

(c)



P(the two scores are the same) $= \frac{6}{36} = \frac{1}{6}$

(d) P(total of 1) $= \frac{0}{36} = 0$

'The two scores are the same' means that the same number is on both dice. So this includes the outcomes (1, 1) (2, 2), (3, 3), etc., i.e. the sums 2, 4, 6, etc.

The lowest possible total of the two dice is 2, so getting a score of 1 is impossible.

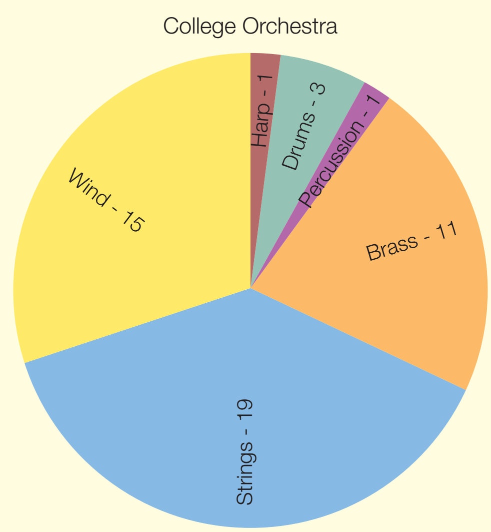
Worked example 10.5

Q. There are 50 people in the college orchestra. Use the pie chart to calculate the probability that:

(a) a person plays a string instrument

(b) a person plays a brass instrument

(c) a person does not play a wind instrument.



continued . . .

There are 50 people in the orchestra, but only 20 play a string instrument. (Note: the harp is a stringed instrument.)

A. (a) $P(\text{string}) = \frac{20}{50}$

There are 50 people in the orchestra, but only 11 play a brass instrument.

(b) $P(\text{brass}) = \frac{11}{50}$

If $\frac{15}{50}$ of the orchestra play a wind instrument, then $1 - \frac{15}{50}$ don't play a wind instrument. (Or you could do $50 - 15 = 35$, out of 50.)

(c) $P(\text{not wind}) = 1 - \frac{15}{50} = \frac{35}{50}$

Exercise 10.3A

1. There are 20 marbles in a bag. 8 of them are red, 7 are green and the remaining 5 are white. Sabel picks one of the marbles out of the bag without looking. Find the probability that the chosen marble is:

(a) red

(b) white or green

(c) not green.

hint

If the marbles are red, white or green, then the event 'white or green' is the same as the event 'not red'.

2. Alfred tosses a coin and rolls a six-sided die. Copy and complete the following table of possible outcomes:

	1	2	3	4	5	6
Head	H1	H2				
Tail					T5	T6

Calculate the probability that Alfred gets:

(a) an even number

(b) a factor of 6

(c) a head and a square number

(d) a tail or a prime number.

3. The letters of the word MATHEMATICS are written on different cards, shuffled and placed face down. One of the cards is picked at random. Calculate the probability that chosen card shows:

(a) an A

(b) an M

(c) not a vowel.

4. Two dice are thrown and the scores are added up. Calculate the following probabilities.

(a) P(total is 11) (b) P(total is at most 10)

(c) P(total is a square number) (d) P(total is a prime number)

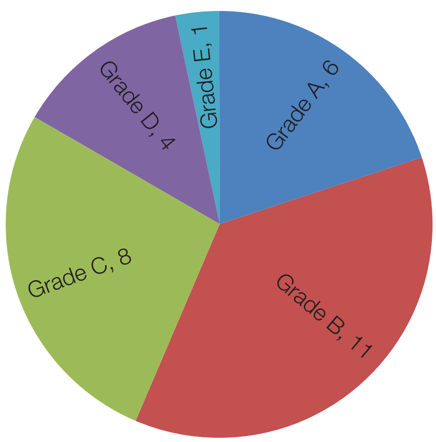
5. A school hockey team plays two matches. Each match could end in a win (W), a loss (L) or a draw (D).

(a) List all the possible outcomes of the two matches. You may find it helpful to copy and complete the table below.

Match 1	W	W	W	L	L	L	D	D	D
Match 2									

(b) What is the probability of drawing one match only?

(c) Calculate the probability of not losing any of the matches.



6. The examination grades of a group of 30 IGCSE students are illustrated on the pie chart.

A student is chosen at random from the group. Calculate the probability that the student scored:

(a) a grade C

(b) a grade B or better

(c) a grade worse than grade C.

7. Ditmar rolls two six-sided dice. He then multiplies the two numbers showing. Complete the following sample space diagram for the outcomes.

	1	**2**	**3**	**4**	**5**	**6**
1						
2		4			10	
3						
4						
5		10			25	
6						36

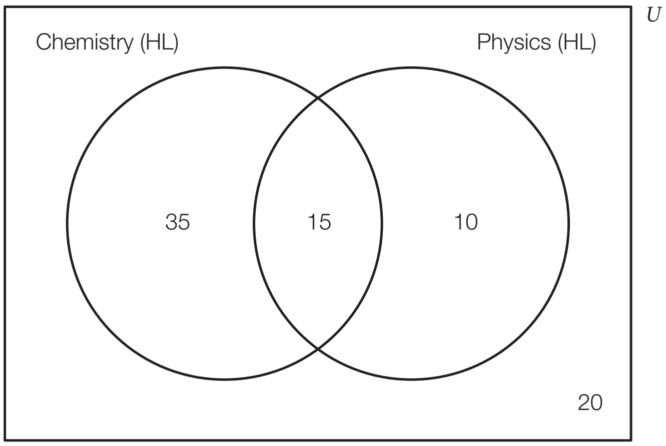
Using Ditmar's results from the sample space diagram, calculate the probability of the product being:

(a) a square number (b) a prime number

(c) a cubed number (d) a multiple of 3

(e) a common multiple of 3 and 5.

8. A survey was carried out on a group of Physics and Chemistry students. The results are shown on the Venn diagram below.



(a) How many students took part in the survey?

(b) If one of the students is chosen at random, calculate the probability that the student:

(i) studies Chemistry (HL)

(ii) studies both Chemistry (HL) and Physics (HL)

(iii) does not study Physics (HL).

9. Three IB students, Andrew, Fareeda and Caitlin, all study History.

(a) Complete the table below to show all the possible combinations of levels of entry, at either Higher (HL) or Standard (SL) level.

Andrew	Fareeda	Caitlin
HL	HL	HL
HL	HL	SL
HL		

(b) What is the total number of possible combinations of choices for the three students?

(c) Find the probability that all three students study History at the same level.

(d) Calculate the probability that exactly two of the students study History at Standard Level.

(e) What is the probability that at least one of the students studies History at Higher Level?

Theoretical and experimental probability

Noah wants to know the probability that he will shoot a hoop successfully in basketball.

Safia wants to know the probability that she will throw two sixes when she throws two dice.

These are two different types of problem and will require different approaches.

The probability of success for Noah depends on how good a player he is and how successful he has been in the past. He calculates the probability **experimentally**, with figures he has collected himself. This is an example of **experimental** probability.

Safia can calculate the probabilities of a particular score on a non-biased die using the theory that all six faces are equally likely to occur $\left(\frac{1}{6}\right)$, so she can calculate her result **theoretically**.

	1	**2**	**3**	**4**	**5**	**6**
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,3	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	**6,6**

Using a sample space diagram, Safia knows that there are 36 different combinations that she should get based on theory if she throws two dice. Only one of these will give her (6,6):

$$P(6,6) = \frac{1}{30}$$

In general, for **theoretical probability**,

$$P(\text{event}) = \frac{\text{number of possible successful events}}{\text{total number of events}}$$

Worked examples 10.1–10.5 are all examples of **theoretical** probability.

But Noah cannot use theory to calculate his chances of success because his chances are **biased**; they depend on his experience and level of skill. He has to rely on his knowledge of past events. Noah has kept a record of his last 80 attempts at getting the ball through a hoop; each of the 80 attempts is called a 'trial'. He was successful 48 times.

In general, for **experimental probability**,

$$P(\text{success}) = \frac{\text{number of successful trials}}{\text{total number of trials}}$$

So for Noah, $P(\text{success}) = \frac{48}{80} = \frac{3}{5}$

Expected value

The expected value is what you would *expect* to get 'on average' within a given distribution based on a given probability. An expected value is a mean calculated from the probability and the number of events/trials.

In general:

$$\text{Expected value} = \text{probability of success} \times \text{number of trials}$$

So, if Noah was to try to score a hoop 10 times, how many times would he *expect* to be successful?

Expected number of scores: $\frac{3}{5} \times 10 = 6$

Noah would expect to score 6 hoops in 10 attempts.

Similarly, if Safia was to throw two dice 12 times, how many times should she expect to throw two sixes?

Expected value $= \frac{1}{36} \times 12 = 0.333333\ldots$

So, we could say (rounding to the nearest whole number) that Sofia would not expect to roll two sixes at all in 12 throws of the dice.

Probability calculations can tell you what to expect but there will always be a variation between the calculated value and the actual value. If you only do a small number of trials, it is unlikely that the expected value will be a good reflection of the actual values because the expected value will be more affected by extremes. The more trials you do, the closer the actual result should be to the expected value.

These are expected values. How true do you think they are? If you throw a fair coin 200 times, do you expect to get exactly 100 heads and 100 tails? Would more throws be more likely to give you exact values?

Worked example 10.6

Q. (a) If Safia throws two dice 180 times, how many times can she expect to throw (6,6)?

(b) If Noah makes 36 attempts to put the ball through the hoop, how many times will he be successful?

In general, expected value = probability × number of trials.

A. (a) $P(6,6) = \frac{1}{36}$

$= \frac{1}{36} \times 180$

$= 5$

Safia should get (6,6) five times.

We know from earlier in this section that the theoretical probability of getting (6,6) is $\frac{1}{36}$. We need to calculate how many times out of 180 throws she is likely to get (6,6). So, we multiply the number of throws by the probability for one throw.

(b) Since $P(\text{success}) = \frac{48}{80} = \frac{3}{5}$

$\frac{3}{5} \times 36 = 21.6$

He can expect to be successful 21.6 (22) times.

We know from earlier in the section that the probability of him scoring a hoop is $\frac{3}{5}$, so multiply this by the number of trials.

Exercise 10.3B

1. Frances tosses two coins. List all the possible outcomes.

   Use your list to work out the probability of getting:

   (a) two heads (b) at least one tail.

   Frances tosses two coins simultaneously 120 times. How many times does she expect to get:

   (c) exactly one tail (d) two heads (e) at least one tail?

2. Emma tosses three coins. List all the possible outcomes.

   Use your list to calculate the probability of getting:

   (a) all heads (b) exactly one tail

   (c) at least one head (d) at most two tails.

   Emma repeats her experiment 96 times. How many times does she expect to get:

   (e) all heads (f) at most two tails?

3. A survey was carried out on a sample of 300 patients in a hospital. The blood groups of the patients are shown in the table below.

Blood group	O	A	B	AB
Frequency	135	120	33	12

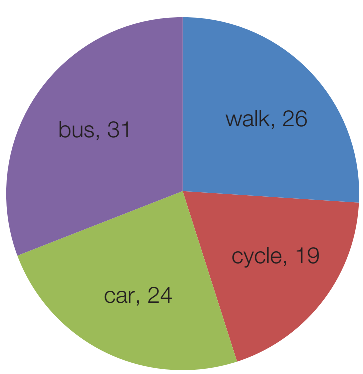
   (a) If a patient is chosen at random, what is the probability that the patient:

   (i) belongs to blood group A?

   (ii) does not belong to blood group O?

   (b) Out of 1300 patients in the hospital, how many of them would you expect to belong to:

   (i) blood group AB? (ii) blood group B?



4. A survey of the regular means of transport to school was carried out on a sample of 100 students in a secondary school.

   (a) If a student is chosen at random, what is the probability that the student:

   (i) travelled to school by car (ii) did not walk to school?

   (b) There are 1100 students in the school. How many do you expect to travel to school:

   (i) by bicycle? (ii) not on foot?

10.4 Mutually exclusive events

Two events are **mutually exclusive** if they cannot happen at the same time.

You cannot toss one coin and get a 'head' **and** a 'tail' at the same time.

You cannot run in the 400 m race with five other people and come both first **and** second in the same race.

These are mutually exclusive events. One **or** the other can happen but **not both** at the same time.

So, coming first in the 400 m race is mutually exclusive to coming second in the same race. This can be written as $P(A \cap B) = \varnothing$ using set notation; there are no elements in the set 'A and B' because A and B cannot happen at the same time. But you could come first *or* second; what is the probability of coming first *or* second? If the probability that you win the 400 m is 0.24 and the probability that you come second is 0.35, then you can calculate the probability that you come either first *or* second by adding the two probabilities together:

$$P(\text{first}) + P(\text{second}) = 0.24 + 0.35 = 0.59$$

This can be written as $P(A \cup B) = P(A) + P(B)$.

We can say that in any general probability situation, if you can add together the probabilities of two events **and** their intersection is an empty set with no elements in it, then the events you are working with are mutually exclusive.

For two mutually exclusive events, you **add** the probabilities because one **or** the other can occur but not both.

RR ***You learned about set notation in Chapter 8.***

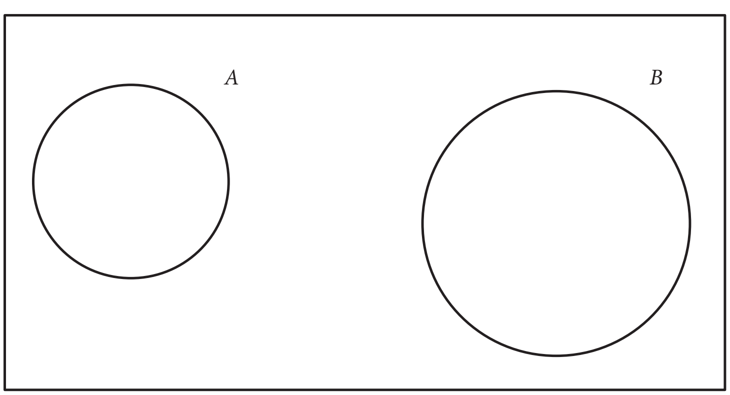
$a = \pi r^2$ In general, if $P(A) + P(B) = P(A \cup B)$, and $P(A \cap B) = \varnothing$, then the events are mutually exclusive.

RR ***Remember from Chapter 8 that in set notation $\cup$ is 'union' ('or'), $\cap$ is 'intersection' ('and'), and $\varnothing$ is the empty set.***

exam tip

Some of the probability formulae are in the Formula booklet, so make sure you know where to look for them. It is important that you rely on your understanding of the probability rather than just on the formulae; this might help you spot an incorrect answer.

The Venn diagram illustrates two mutually exclusive events; there is no overlap between the sets.



Exercise 10.4

1. A jar contains 9 red, 12 green and 11 yellow marbles. A marble is chosen at random. What is the probability of choosing:

   (a) a green marble? (b) a yellow or red marble?

2. There are 7 apples, 3 plums and 6 pears in a fruit bowl. Gemma picks a fruit randomly from the bowl. Determine the probability of Gemma picking:

   (a) a pear

   (b) a plum

   (c) a pear or a plum.

3. There are 24 cars at a car park. 6 of them are estate cars and 4 of them coupés. One of the owners has just reported to the car park attendant that her car's wing mirror has been damaged.

   Calculate the probability that the damaged car:

   (a) is an estate car (b) is a coupé car

   (c) is an estate or coupé (d) not an estate car.

4. A school Lacrosse team is playing in a friendly match. The probability of winning is 0.6 and the probability of drawing is 0.25.

   (a) Find the probability of losing the match.

   (b) Work out the probability of winning or drawing.

   (c) Calculate the probability of not drawing the match.

5. There are 28 IB students working in a school library. 5 of them study Mathematics (HL) and 14 of them study Mathematics (SL).

   One of the students is using the photocopier. What is the probability that the student does:

   (a) Mathematics (HL)?

   (b) Mathematics (HL) or Mathematics (SL)?

   (c) neither Mathematics (HL) nor Mathematics (SL)?

6. The grade distribution of GCSE results for a secondary school in the UK for 2011 is shown below.

Grade	A*	A	B	C	D	E	F	G	U
% of entries	7.8	15.4	21.7	24.9	15.1	7.8	4.1	2.0	

   (a) Given that a U grade is a fail, find the percentage of entries which were failures.

(b) If one of the examination entries is selected at random, find the probability that it earned:

(i) a C grade or higher

(ii) an E grade or lower

(iii) an A or A*.

(c) The school registered 2000 entries in total. Calculate the expected number of:

(i) A or A* grades (ii) A* to C grades.

10.5 Probability of combined events

Combined events are those that are not mutually exclusive; they are events that can happen at the same time or have some overlap. In these situations you can find the probability of getting '*A* or *B*' or '*A* and *B*'. The general formula for combined events is:

$a=\pi r^2$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

RR ***Remember from Chapter 8 that in set notation ∪ is 'union' ('or') and ∩ is 'intersection' ('and').***

exam tip

This is the formula for combined events, but remember that a good diagram can help you to understand a situation and complete the calculations without using the formula.

This formula is in the Formula booklet, but in examinations questions using formulae on their own will not be set. All questions will be set in context.

A marketing company is doing a household survey. They survey 42 houses and find: 23 households with gas cookers, 17 households with electric cookers and five who use dual fuel cookers (cookers that use both gas and electricity). Seven other houses cook on oil-fired cookers. What is the probability that a household cooks with gas or electricity or both?

Using the formula, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$:

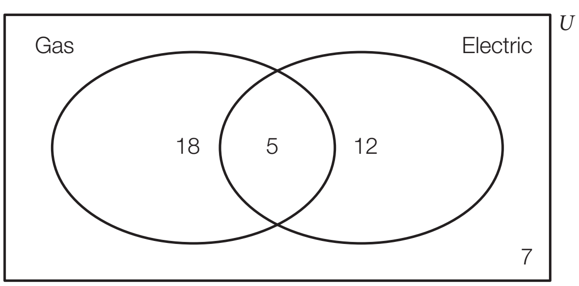
If G represents gas and E represents electricity, then

$$P(G \cup E) = P(G) + P(E) - P(G \cap E)$$

From the original information, you can calculate that

$$P(G \cup E) = \frac{23}{42} + \frac{17}{42} - \frac{5}{42} = \frac{35}{42}$$

You might find this easier to visualise using a Venn diagram. The five dual fuel cookers go into the intersection (gas **and** electricity).



Households that cook with gas or electricity or both are in the **union** of 'gas', 'electricity' and 'gas and electricity'. Since $18 + 5 + 12 = 35$, there are 35 households cooking with gas or electricity or both, so $P(G \cup E) = \frac{35}{42}$.

Independent events

Two events are **independent** if the first one has no influence on the second.

The probability of getting a head when you throw a coin is $\frac{1}{2}$. What is the probability of getting a tail if you throw it again? It is still $\frac{1}{2}$, as the two events do not affect each other. They are independent. The probability of throwing a head **and** then another head can be calculated as follows:

$$P(\text{two heads}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

For independent events, you **multiply** the probabilities of each event because you could have one **and** the other.

$a = \pi r^2$ In general, for independent events, $P(A \cap B) = P(A) \times P(B)$.

The formula can be used to test whether or not events are independent.

Let U = {natural numbers from one to twenty}

A = {multiples of 5}

B = [multiples of 4}

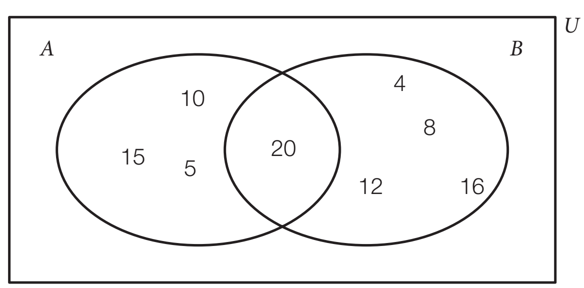
Are A and B independent?

You can check by testing the probabilities.

A = {5, 10, 15, 20}

B = {4, 8, 12, 16, 20}

Only the number 20 occurs in both lists:



$$P(A) = \frac{4}{20}, P(B) = \frac{5}{20} \text{ and } P(A \cap B) = \frac{1}{20}$$

$$P(A) \times P(B) = \frac{4}{20} \times \frac{5}{20} = \frac{1}{20} = P(A \cap B)$$

A and B are proved to be independent.

Worked example 10.7

Q. In a local art competition, the probability that Barbara wins a prize for her design is 0.4 and the probability that Alan wins a prize for his photograph is 0.58. The probability that they both win a prize is 0.25.

(a) Find $P(A \cup B)$, the probability that either Alan or Barbara wins a prize.

(b) Are the events A and B independent? Explain your answer.

A. (a) $P(A \cup B) = 0.4 + 0.58 - 0.25 = 0.73$

A 'or' B means 'add' the probabilities. The events are combined, so use the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) For events to be independent,

$P(A) \times P(B) = P(A \cap B)$.

$P(A) \times P(B) = 0.4 \times 0.58 = 0.232$

$P(A \cap B) = 0.25$ and $P(A) \times P(B) = 0.232$

so the events are not independent.

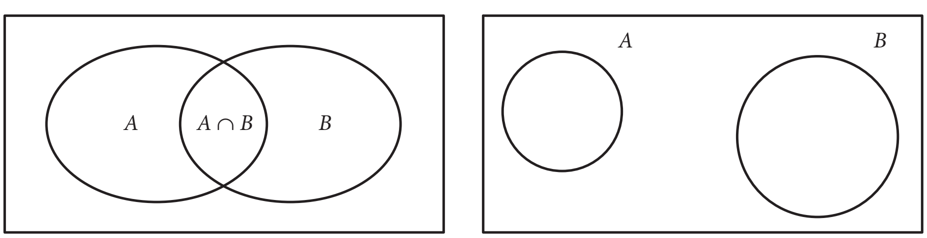
$P(A \cap B) = 0.25$ is given to you in the question as the probability that they both win a prize. The calculation $P(A) \times P(B)$ gives a different answer, so A and B are not independent.

Can events be mutually exclusive and independent?

When you read a problem, you need to be quite clear that you understand the difference between mutually exclusive and independent events.

Mutually exclusive events cannot be independent. Look at the Venn diagrams and the formulae below.

Independent events **Mutually exclusive events**



$P(A \cap B) = P(A) \times P(B)$

$P(A) + P(B) = P(A \cup B)$, and $P(A \cap B) = 0$

If A and B are mutually exclusive, then $P(A \cap B) = 0$. If they are independent and both occur with positive probability, then $P(A \cap B) = P(A) \times P(B) \neq 0$. Since $P(A \cap B)$ cannot be zero and non-zero at the same time, two events cannot be mutually exclusive and independent at the same time.

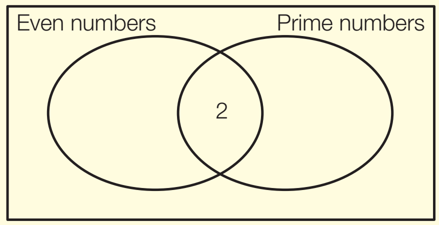
Worked example 10.8

Q. The numbers 2, 3, 4, 5, 6, 7, 8 and 10 are written out on eight separate pieces of paper. One number is chosen.

(a) Are the events 'choosing a 2' and 'choosing an odd number' mutually exclusive?

(b) Are the events 'choosing a 2' and 'choosing a prime number' mutually exclusive?

(c) Are the events 'choosing an even number' and 'choosing a prime number' independent?



A. (a) 2 is an even number; if only one number is picked then an odd and an even number cannot be chosen at the same time, so these events are mutually exclusive.

(b) 2 is an even number and also a prime number, so choosing a 2 and choosing a prime number are not mutually exclusive events.

(c) If $E = \{2, 4, 6, 8, 10\}$ and $P = \{2, 3, 5, 7\}$:

$$\mathrm{P}(E) = \frac{5}{8},\ \mathrm{P}(P) = \frac{4}{8} \text{ and } \mathrm{P}(E \cap P) = \frac{1}{8}$$

$$\mathrm{P}(E) \times \mathrm{P}(P) = \frac{5}{8} \times \frac{4}{8} = \frac{5}{16} \neq \frac{1}{8}$$

The events are not independent

Exercise 10.5

1. For the two events A and B, $\mathrm{P}(A \cup B) = 0.61$, $\mathrm{P}(A) = 0.33$, $\mathrm{P}(B) = 0.82$.

   (a) Find $\mathrm{P}(A \cap B)$.

   (b) Explain with reasons whether the events A and B are independent.

2. The events A and B are mutually exclusive. $\mathrm{P}(A) = 0.57$ and $\mathrm{P}(B) = 0.26$. Find $\mathrm{P}(A \cup B)$.

3. Gemma is ready for her driving test. The probability of her passing first time is 0.65. If she fails her first test, the probability of her passing on the second attempt is 0.85.

   (a) Calculate the probability of Gemma passing in exactly two attempts.

   (b) Find the probability of Gemma passing within two attempts.

4. Out of the 260 registered members of a golf club, 200 of them are male. 31 of the members are aged 65 or older. 55 of the female members are aged under 65.

   (a) If one of the golfers is selected at random, what is the probability that he or she is:

      (i) a male golfer under 65 years of age?

      (ii) a female golfer who is 65 or older?

   (b) Two of the golfers won awards in the last season. Calculate the probability that:

      (i) at least one of them is a female golfer and younger than 65

      (ii) neither of them is a male golfer older than 65.

5. Tim and Gary have carried out a survey of students resident in their neighbourhood. The number of students enrolled in different levels of education is summarised in the table below.

	Elementary School	**High School**	**College**
Frequency	370	170	190

   Three students from the neighbourhood have been nominated to be given awards for academic excellence and service in the local community.

   Calculate the probability that:

   (a) all three students are in High School

   (b) exactly two of the students are College students

   (c) at least one of the students is in Elementary School.

6. Jonah tosses a coin and rolls a six-sided die. Find the probability of Jonah getting:

   (a) a head on the coin and an even number on the die

   (b) a tail on the coin and a factor of 12 on the die.

7. Diego manages a football team. His team is playing a match. The team consists of a goalkeeper, four defenders, four midfield players and two strikers. At half time Diego decides to make two substitutions. If all the players have the same chances of being substituted, calculate the probability that:

   (a) the players to be substituted are not strikers

   (b) at least one striker is being substituted

   (c) at most one midfielder is being substituted

   (d) the goalkeeper is not being substituted.

10.6 Tree diagrams and Venn diagrams

Many probability questions can be solved by using tree diagrams or Venn diagrams. Both give a good picture of the information in a question and help you to move from there to a solution. In some problems it will be clear which diagram to choose; in others you may find that you can use either and it will depend on your personal preference. It is important to make sure that you are confident in using both and not afraid to move from one type of diagram to the other if it allows you to work through the original problem more easily.

Using tree diagrams

Tree diagrams allow you to see all the possible outcomes of successive events, and to calculate the probability of those events.

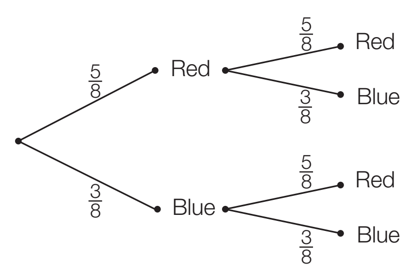
FF ***They are very useful in section 10.7.***

Priya has a bag containing 5 red balls and 3 blue ones. She picks out a ball, notes the colour and replaces the ball. She then takes out a second ball, notes the colour and puts the ball back in the bag.

Priya is doing two successive actions. The first action can be represented by the first two branches of the tree, and her second action can be represented by the branches that follow on from the first ones.

Using theoretical probability, Priya knows that P(red) = $\frac{5}{8}$ and P(blue) = $\frac{3}{8}$.

She can build her tree diagram like this:



exam tip

Remember to **multiply** across the branches, and **add** down the branches.

The diagram shows all the possible combinations and allows Priya to calculate the probabilities.

$$\text{P(red, red)} = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$$

Red is followed by another red ('and'), so Priya has to *multiply.*

$$\text{P(red, blue)} = \frac{5}{8} \times \frac{3}{8} = \frac{15}{64}$$

$$\text{P(blue, red)} = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$$

$$\text{P(blue, blue)} = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$$

$$\text{P(two different colours)} = \frac{5}{8}\times\frac{3}{8}+\frac{3}{8}\times\frac{5}{8}=\frac{25}{64}$$

Priya can check that she has covered every combination:

$$\frac{25}{64}+\frac{15}{64}+\frac{15}{64}+\cdots+\frac{9}{64}=1$$

Two colours could be red first then blue (R **and** B) **or** blue first then red (B **and** R). '**And**' means that 'R and B' need to be multiplied, and 'B and R' need to be multiplied. '**Or**' indicates that you can't have both (R, B) and (B, R) at the same time, so they are mutually exclusive and their product probabilities must be *added.*

RR ***Remember from section 10.1 that all probabilities of an experiment add up to 1.***

The branches of a tree diagram do not have to show the same values.

Worked example 10.9

Q. If the weather is fine, Mr Liu walks to work. The probability that he is late is 0.1.

If it rains, Mr Liu catches the bus and the probability that he is late is 0.3.

The probability of rain is 0.45.

(a) What is the probability that Mr Liu is late for work?

(b) From the tree diagram, calculate the probability that:

(i) Mr Liu catches the bus and is not late.

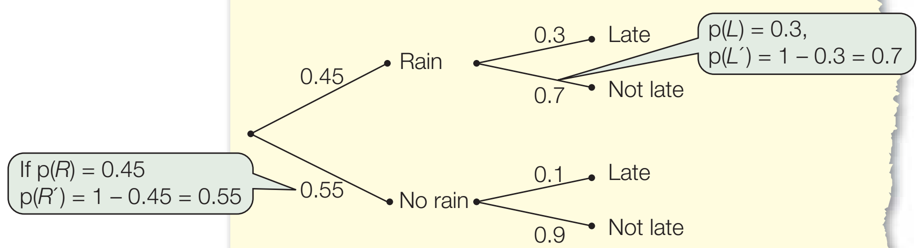
(ii) Mr Liu walks to work and is late.

(iii) Mr Liu is not late for work.

(c) Mr Liu works 255 days a year. On how many days does he expect **not** to be late?

In this situation, Mr Liu makes a decision dependent on the weather. So the first event is 'rain' or 'no rain' and the second event is 'late' or 'not late' depending on the outcome of the first event.

continued . . .



There are two ways that he can be late, if he travels by bus **or** if he walks.

Using the tree diagram: follow the branches that show it rains and he is late.

A. (a) $P(R, L) = 0.45 \times 0.3 = 0.135$

Using the tree diagram follow the branches that show it does not rain and he is late.

$P(R', L) = 0.55 \times 0.1 = 0.055$

It cannot rain **and** not rain, but it can rain **or** not rain, so the events are mutually exclusive and the probabilities are added.

$P(L) = 0.135 + 0.055 = 0.19$

This is the probability that it rains (he catches the bus) and he is not late. Follow the appropriate branches and multiply the probabilities.

(b) (i) $P(R, L') = 0.45 \times 0.7 = 0.315$

This is the probability that it does not rain (he walks to work) and he is late. Multiply the probabilities along the branch.

(ii) $P(R', L) = 0.55 \times 0.1 = 0.055$

‘Late’ and ‘Not late’ are complementary events so you can use your answer to part (a) here.

(iii) $P(L') = 1 - 0.19 = 0.81$

Use the probability from part (b)(iii).

(c) $P(L') = 0.81$

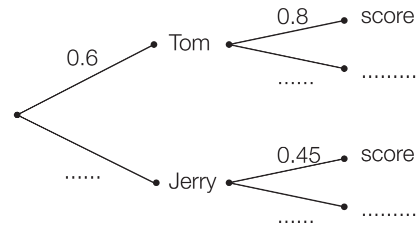
Expected value = frequency × probability.

$255 \times 0.81 \approx 207$

Mr Liu expects not to be late about 207 days of the year.

Exercise 10.6A

1. Tom and Jerry play for the same hockey team. The manager always selects either Tom or Jerry for the team. The probability of Tom being selected is 0.6. If he is selected, the probability of Tom scoring is 0.8. The probability of Jerry scoring is 0.45.



   (a) Copy and complete the tree diagram.

   (b) For the next match, calculate the probability of:

      (i) Tom playing and scoring

      (ii) either Tom or Jerry scoring

      (iii) neither Tom nor Jerry scoring.

   (c) If Jerry plays in the next two successive matches, calculate the probability of:

      (i) scoring in just one of the matches

      (ii) scoring in at most one of the matches.

2. Kofi and Kojo are playing a game of tennis. The probability of Kofi serving an ace is 0.6. The probability of Kojo serving an ace is 0.55.

   Calculate the probability that:

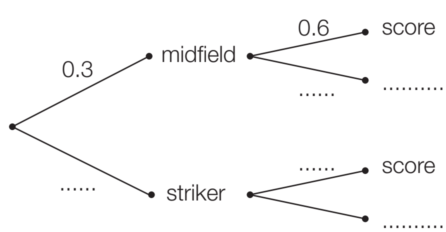
   (a) both Kofi and Kojo serve aces

   (b) neither Kofi nor Kojo serves an ace

   (c) at least one of them serves an ace.

3. Leslie plays football. He plays either in midfield or as a striker. The probability of his playing in midfield is 0.3. When he plays in midfield the probability of scoring is 0.6. When he plays as a striker the probability of scoring is 0.9.

   (a) Copy and complete the tree diagram below.



   If Leslie plays one match, calculate the probability of him:

   (b) scoring as a midfielder

   (c) scoring

   (d) playing as a striker and not scoring

   (e) not scoring.

4. Julius takes two penalty kicks in a game of rugby. For each kick the probability of scoring is equal to 0.85.

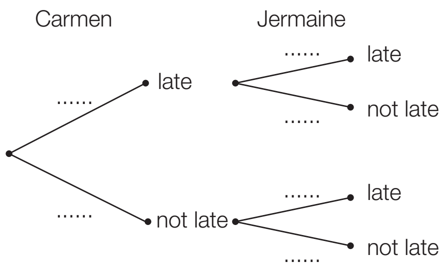
   Draw a tree diagram, and hence calculate the probability of Julius:

   (a) missing both kicks

   (b) scoring once

   (c) missing no more than once.

5. Kye has invited two friends, Carmen and Jermaine, to his birthday party in Milton Keynes. Carmen is travelling by car and Jermaine by train. The probability of Carmen arriving late is 0.34, while the probability of Jermaine being late is 0.28.



   (a) Copy and complete the tree diagram.

   (b) Calculate the probability that:

   (i) neither of the friends is late

   (ii) both friends are late

   (iii) only Carmen is late

   (iv) only one of the friends is late.

Using Venn diagrams

In some situations it is easier to display the information you are given in a Venn diagram.

Worked example 10.10

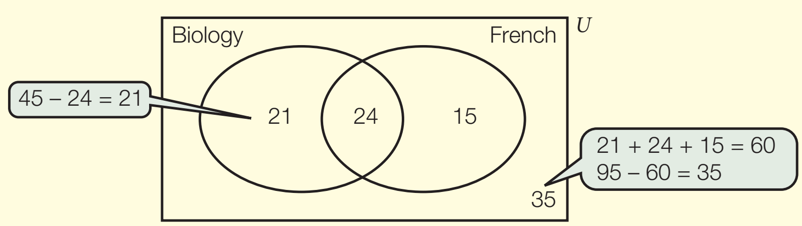
Q. A school has 95 students who are taking their IB diploma this year. There are 45 students who study Biology and 39 who study French. There are 24 students who study both Biology and French. Draw a Venn diagram and calculate the probability that:

(a) a student studies Biology or French

(b) a student does not study Biology

(c) a student does not study Biology or French.

A.



The Venn diagram allows you to use the fact that 24 students are studying both subjects.

continued . . .

From the diagram:

'*B* or *F*' means the union of *B* and *F*, so

21 + 24 + 15 = 60

(a) $P(B \text{ or } F) = \frac{60}{95}$

'Not *B*' means everywhere in the diagram except *B*: 15 + 35 = 50

(b) $P(B') = \frac{50}{95}$

'Not *F* or *B*' means everywhere that is not *F* or not *B*. In this case it is outside both circles.

(c) $P(F' \text{ or } B') = \frac{35}{95} = \frac{7}{19}$

You can keep your answer as $\frac{35}{95}$ or write the fraction in its simplest form.

Exercise 10.6B

1. The sets *A* and *B* are subsets of *U*. They are defined as follows:

U = {prime numbers less than 20}

A = {prime factors of 30}

B = {prime factors of 66}

(a) Draw a Venn diagram to represent the relationship between sets *U*, *A* and *B*. List the elements of the sets in the corresponding regions of the Venn diagram.

(b) Find the probability that a number chosen randomly from the universal set *U* is:

(i) a factor of 55

(ii) a factor of 39.

hint

The prime factors of 12 are {2, 3}

2. In a class of 24 students, 9 study History and 14 study Economics. Four of the students study neither History nor Economics.

(a) Draw a Venn diagram to represent the information given above.

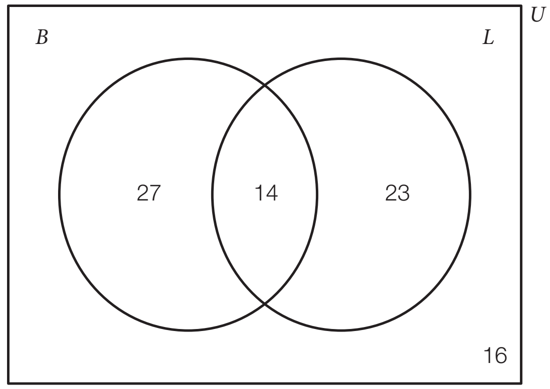
(b) If a student is chosen at random, calculate the probability that the student:

(i) studies History or Economics

(ii) does not study Economics

(iii) studies Economics or History, but not both.

3. The Venn diagram below shows how many members of a Leisure Club are interested in bowling and line dancing.



U = {all members of the club}

B = {members who enjoy bowling}

L = {members who enjoy line dancing}

(a) How many members does the club have?

(b) If one of the members decides to leave the club, what is the probability that she/he:

(i) enjoys line dancing

(ii) enjoys line dancing but not bowling

(iii) does not enjoy bowling

(iv) enjoys line dancing or bowling, but not both

(v) enjoys both line dancing and bowling, or neither?

10.7 Probability 'with replacement' and 'without replacement'

If a problem is in two stages, look carefully at the explanation.

Maryam has a bag of sweets. There are 7 red sweets and 8 green sweets. Without looking in the bag, Maryam picks out two sweets, one at a time. She has two choices once she has picked out her first sweet:

1. She can put the first sweet back in the bag before taking the second one; this is a problem 'with replacement' and the probabilities of selecting a sweet are the same for the first sweet and the second sweet.
2. She can eat the first sweet, and then take the second sweet. This has altered the probabilities for the second stage and is a problem 'without replacement'. The probability of selecting a sweet for the second time is different from the probability of selecting the first sweet.

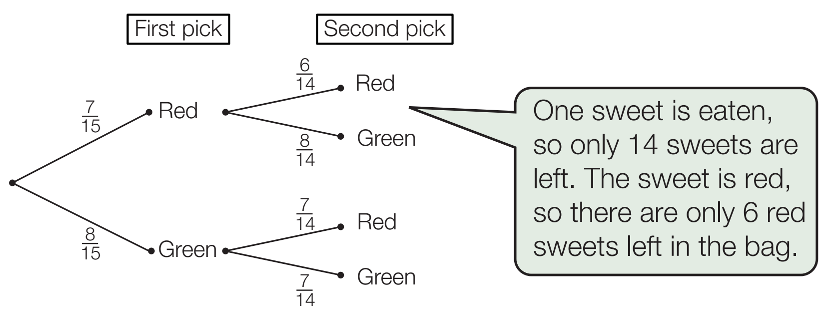
Without replacement

Maryam has a bag of sweets. There are 7 red sweets and 8 green sweets. Without looking in the bag, Maryam takes out a sweet and eats it. She then takes another sweet and eats that.

(a) What is the probability that she eats two green sweets?

(b) What is the probability that she eats two different coloured sweets?

This problem is similar to the example in section 10.6 where Priya is picking coloured balls, so the best diagram to use is a tree diagram. However, Maryam has *eaten* the first sweet instead of *replacing* it. The number of sweets in the bag is now different; therefore the probabilities will be different when she takes a second sweet.



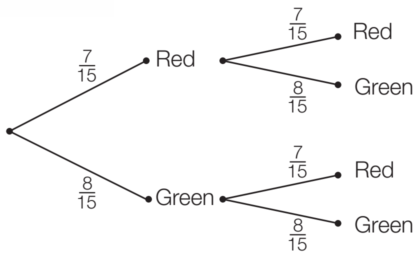
Using the tree diagram:

(a) $P(G, G) = \frac{8}{15} \times \frac{7}{14} = \frac{4}{15}$

There are 15 sweets and 8 are green so the probability of picking a green sweet is $\frac{8}{15}$. After one green sweet is eaten, there are only 14 sweets left in total and only 7 green sweets, so the probability of picking a green sweet has changed to $\frac{7}{14}$.

(b) $P(R, G \text{ or } G, R) = \frac{7}{15} \times \frac{8}{14} + \frac{8}{15} \times \frac{7}{14} = \frac{8}{15}$

There are two ways to pick two different colours: red then green, **or** green then red.



With replacement

If Maryam does not eat the first sweet but puts it back in the bag, the problem is different. This is now a probability question 'with replacement' so the probabilities on the tree diagram do not change.

Using the tree diagram the probabilities are now:

(a) $P(G, G) = \frac{8}{15} \times \frac{8}{15} = \frac{64}{225}$

(b) $P(R, G \text{ or } G, R) = \frac{7}{15} \times \frac{8}{15} + \frac{8}{15} \times \frac{7}{15} = \frac{112}{225}$

Exercise 10.7

1. A bag contains 9 yellow marbles and 4 blue marbles. A marble is taken from the bag and its colour noted but it is **not** replaced. Another marble is then taken. Calculate the probability that:

(a) at least one of the marbles is yellow

(b) the marbles are of different colours

(c) neither marble is blue.

2. A bag contains 12 red marbles and 8 green marbles. A marble is taken from the bag and its colour noted. It is then replaced. Another marble is then taken and its colour noted. Calculate the probability that:

(a) both marbles are green

(b) the marbles are of different colours

3. There are 18 students in a class, 8 boys and 10 girls. A teacher asks for two volunteers to hand out some books. Calculate the probability that:

(a) both students are boys

(b) neither of the students is a boy

(c) at least one of the students is a girl.

4. A jar contains 7 red marbles, 5 green marbles and 3 blue marbles. Alice takes a marble, notes its colour and puts it back in the jar. She then takes another marble. Calculate the probability that:

(a) the marbles are of the same colour

(b) the marbles are of different colours

(c) there is at least one blue marble.

5. A bag contains 12 soft-centred sweets and 8 hard-centred ones. Without looking, Robert takes a sweet from the bag, eats it and then takes another one. Calculate the probability that:

   (a) neither of the sweets is soft-centred

   (b) not more than one of the sweets is hard-centred

   (c) there is at least one of each type of sweet.

6. The table below shows the breakdown of the examination results in Mathematical Studies for an IB College.

Level	3	4	5	6	7
Frequency	2	8	12	7	3

   If one student is chosen at random, calculate the probability that the student achieved:

   (a) a Level 6 or 7

   (b) a Level 4 or higher.

   Two students are selected randomly. With the aid of a tree diagram, calculate the probability that:

   (c) both of them achieved Level 7s

   (d) just one of them achieved a Level 7

   (e) at least one of them achieved a Level 7.

7. Terry likes playing snooker. The values of the seven coloured balls are as follows:

Colour	red	yellow	green	brown	blue	pink	black
Points value	1	2	3	4	5	6	7

   Terry picks two of the snooker balls randomly and calculates the total points score. Neither ball is replaced.

   (a) Draw a sample space diagram to show the list of possible sums.

   (b) Calculate the probability of picking two balls with a total points score:

      (i) equal to 8

      (ii) not greater than 10

      (iii) not less than 12.

→

8. The table below shows the orders placed by a group of 140 passengers in an airport café.

	Bagel	Croissant	Toast	Total
Tea	26	18	33	
Coffee	34	12	17	
Total				

(a) A catering officer asks one of the customers what they have just ordered. What is the probability that the customer:

(i) did not order a croissant

(ii) ordered tea

(iii) ordered coffee but not toast?

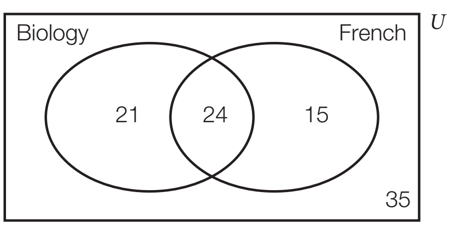
(b) Mr and Mrs Knight have just finished their meal at the café. What is the probability that:

(i) they both had coffee and a bagel

(ii) neither of them had toast

(iii) at least one of them had a croissant?

10.8 Conditional probability



Look at this Venn diagram again:

The diagram shows that 39 out of 95 students are taking French, so:

$$\mathrm{P(French)} = \frac{39}{95}$$

But suppose that you ask a slightly different question.

'Given that a student is studying French, what is the probability that they are also studying Biology?' The extra information 'given that' alters the problem; it is an example of **conditional probability**, where you are given some additional information that you need to use in your calculation.

The word 'given' in the question tells you that you are solving a conditional probability question.

Look at the Venn diagram again. The probability is no longer calculated from all the students at the college; you calculate the probability using the number of students who are studying French as the total. 39 are studying French, and 24 students are studying French and Biology. So, the probability that a student studies Biology given that they study French:

$$\mathrm{P}(B \mid F) = \frac{24}{39}$$

'$B \mid F$' means 'B given F'

In general terms we can say that the probability of A given B is equal to the probability of 'A and B' divided by the probability of B. The general formula for conditional probability is shown below.

$a = \pi r^2$

$$\text{In general, } P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

You are asked about another friend. What is the probability that he is studying French given that he is **not** studying Biology?

You can approach this in two ways:

1. Look at the Venn diagram. 15 students study only French, and 50 students (15 + 35) do not study Biology, so $P(F \mid B') = \frac{15}{50}$.
2. Or you can use the formula:

$$P(F \mid B') = \frac{P(\text{French and no Biology})}{P(\text{no Biology})} = \frac{\frac{15}{95}}{\frac{50}{95}} = \frac{15}{50}$$

In both cases: $P(F \mid B') = \dfrac{15}{50}$

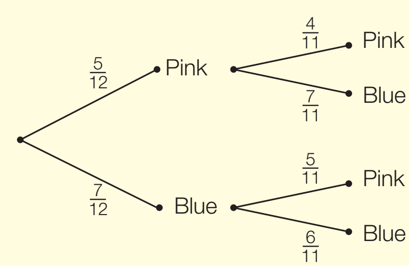
exam tip

Make sure you remember that in conditional probability the denominator will change depending on the conditions. Focus on the words instead of the formula.

Worked example 10.11

Q. Zoe has a drawer containing 5 pink socks and 7 blue socks. Without looking, she picks out a sock, puts it on her bed and takes out another sock.

The tree diagram looks like this:



(a) What is the probability that Zoe picks out two socks of the same colour?

(b) Given that she picks out two socks that are the same colour, what is the probability that they are both blue?

A. (a) $P(\text{same colour}) = \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{6}{11} = \frac{31}{66}$

She could pick out (pink, pink) ($\frac{5}{12} \times \frac{4}{11}$) **or** (blue, blue) ($\frac{7}{12} \times \frac{6}{11}$). Remember that once the first sock has been picked, there are only 11 socks left and one less of the selected colour.

The word 'given' in the question indicates that the probability is conditional. Use the general formula $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ and substitute in the appropriate values obtained from part (a).

continued . . .

(b) $P(\text{both blue} \mid \text{same colour}) = \frac{P(\text{both blue})}{P(\text{same colour})}$

$P(\text{both blue}) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$

$P(\text{same colour}) = \frac{31}{66}$ (from part (a))

$P(\text{both blue} \mid \text{same colour}) = \frac{7}{22} \div \frac{31}{66} = \frac{21}{31}$

exam tip

Answering a conditional probability question involves reading the question carefully, and then creating a good sample space diagram. If you do this successfully, you may not need to use the formula.

Worked example 10.12

Q. This table shows the hair colour of 64 children at a nursery.

	Brown	Blonde	Black
Boy	10	15	7
Girl	8	19	5

Use the sample space diagram to answer the following questions; you do not need to use the formula.

(a) What is the probability that a child has blonde hair?

(b) Given that a child has blonde hair, what is the probability that it is a girl?

Using the table, the total frequency of blonde is 34 (15 + 19).

A. (a) $P(\text{Blonde}) = \frac{15+19}{64} = \frac{34}{64} = \frac{17}{32}$

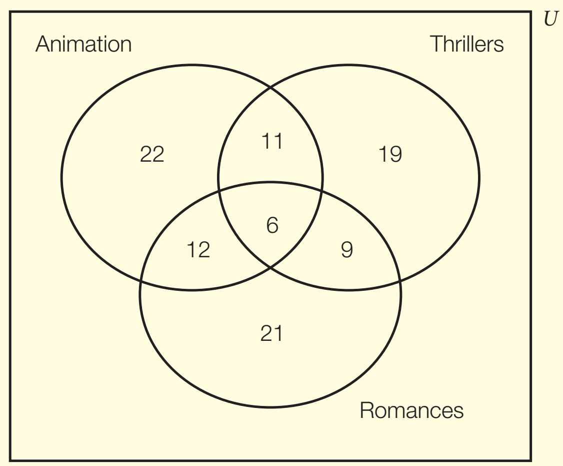
There are 19 blonde girls and a total of 34 blonde children.

(b) $P(\text{Girl} \mid \text{Blonde}) = \frac{19}{34}$

Worked example 10.13

Q. 100 members of a film club are asked which type of film they like watching: animation, thrillers or romances.

The results are given on this Venn diagram. Only six members like all three genres.



(a) What is the probability that someone likes watching animated films?

(b) Given that they like watching animated films, what is the probability that they also enjoy romances? Do not use the formula.

Using the Venn diagram: $22 + 11 + 12 + 6 = 51$ people out of 100 like animated films.

A. (a) $P(\text{animation}) = \frac{51}{100}$

Using the Venn diagram: of the 51 people who like animations, $12 + 6 = 18$ people also like romances.

(b) $P(R \mid A) = \frac{18}{51}$

Worked example 10.14

Q. The results of a conditional probability calculation can be unexpected.

Suppose that there is a disease that affects one person in every thousand. It is curable but the treatment is unpleasant. A test has been developed that gives a positive result for the presence of the disease in 97% of the people who have the disease and 4% of those who do not. Is it a reliable test?

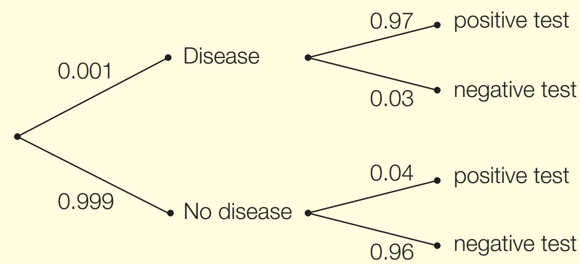
If you were given these figures by a doctor, would you take the test? After looking at the mathematics on the next page, would you change your mind?

continued . . .

$P(\text{disease}) = \frac{1}{1000} = 0.001$

$P(\text{no disease}) = 1 - 0.001 = 0.999$

A. This is the tree diagram:



$P(\text{positive test}) = 0.001 \times 0.97 + 0.999 \times 0.04 = 0.04093$

$P(\text{disease} \mid \text{positive test}) = \frac{0.001 \times 0.97}{0.04093} = 0.0237$ or 2.37%

There is a 2.37% chance that someone with a positive test result has the disease. The test needs to be much more reliable than this if it is to be useful.

Exercise 10.8

1. The following table shows the number of IB students studying Physics and Economics in a College. It is also known that none of the Physics students studies Economics.

	Physics	Economics
Standard Level	28	46
Higher Level	12	14

(a) One of the students is chosen at random, calculate the probability that the student studies:

(i) Physics or Economics at Higher Level

(ii) at Higher Level, given that the student studies Physics.

(b) It is known that Anna studies Economics. What is the probability that she studies Economics at Standard Level?

2. Out of 100 students at a Language School, 43 study Russian and 59 study Chinese. 17 of the students study neither Russian nor Chinese.

(a) Represent the given information on a fully labelled Venn diagram.

(b) How many of the students study both Russian and Chinese?

(c) A student is chosen at random from the school. Calculate the probability that the student studies:

(i) both Chinese and Russian

(ii) Chinese, given that he or she does not study Russian

(iii) Russian, given that he or she studies Chinese.

3. The following table shows the results of a survey carried out among all the Year 12 students in a school. The table shows the number of students who had eaten breakfast at home or had lunch at school on the last day of term.

		Breakfast	
		Yes	**No**
School lunch	**Yes**	42	65
	No	25	18

Find the probability that a Year 12 student, chosen at random on that day:

(a) had breakfast and school lunch

(b) had school lunch, given that they had breakfast

(c) did not have breakfast, given that they did not have school lunch.

Summary

You should know:

- the basic principles of probability
  - a certain result has a probability of 1
  - an impossible result has a probability of 0
  - all probabilities lie between 0 and 1
  - the results of a probability experiment are called outcomes and an individual outcome being investigated is an event
  - the probabilities of all possible outcomes of a probability experiment add up to 1
  - if an event A' is complementary to A, their probabilities add up to 1 ($P(A') = 1 - P(A)$).

continued ...

- what a sample space is and how to construct sample space diagrams
- how to calculate theoretical and experimental probabilities, and that the general formula for calculating probability is $P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$.
- what the expected value is and how to calculate it
- how to recognise and calculate the probability of combined, mutually exclusive and independent events
- how to use tree diagrams, Venn diagrams, sample space diagrams and tables of outcomes to calculate probabilities
- that probability 'with replacement' is not the same as probability 'without replacement'
- how to calculate conditional probability using sample space diagrams (and the formula as well).

Mixed examination practice

Exam-style questions

1. The sample space diagram for rolling a **biased** six-sided die is shown below.

x	1	2	3	4	5	6
P(x)	a	0.3	0.2	0.1	$2a$	a

(a) Work out the value of a.

(b) Calculate the probability of rolling a 5.

(c) Find the probability of not rolling a 5.

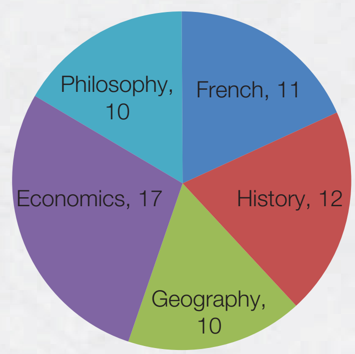
2. One letter is chosen at random from the word CALCULUS. What is the probability of choosing the letters L or C?

3. The letters of the word BOTANICAL are written on nine separate cards and placed on a table. Nikolai picks one of the cards at random.

Calculate the probability he picks a card with the letter:

(a) A (b) T (c) A or T.

4. 60 IB students were asked to name their favourite subjects. The pie chart illustrates the responses given by the students.



If one of the students is picked at random, find the probability that the student chose:

(a) Economics

(b) French or History

(c) neither Geography nor Philosophy.

5. Out of 80 university students, 48 of them were over 18 years old and 32 of them had cars. 20 of the over-18-year-olds had cars.

(a) Complete the following table based on the information from above.

	Over 18's	**18 or under**	**Total**
Have cars			
Do not have cars			
Total			80

One of the 80 students is chosen at random. What is the probability that the student:

(b) is not over 18 years old and owns a car

(c) does not own a car?

6. Twin sisters, Wing and Nam, are entered for the Senior Mathematics Challenge. They could each win a Gold Certificate, a Silver Certificate, a Bronze Certificate or nothing. Given that each of them wins a certificate, complete a full sample space diagram for the possible outcomes.

Wing	Gold	Gold							
Nam	Gold								

(a) What is the total number of possible outcomes?

(b) What is the probability of:

(i) both sisters winning bronze certificates?

(ii) neither of them winning a gold certificate?

(iii) the sisters winning different certificates?

7. A survey of 140 football fans was carried out at the Stadio Olimpico in Rome. They were asked whether they lived in Rome and, if they did, which football team they supported.

$U = \{\text{football fans}\}$
$R = \{\text{residents of Rome}\}$
$L = \{\text{Lazio fans}\}$
$n(R \cup L) = 134$
$n(R) = 110$
$n(R \cap L) = 42$

(a) Draw a Venn diagram to represent the information from above.

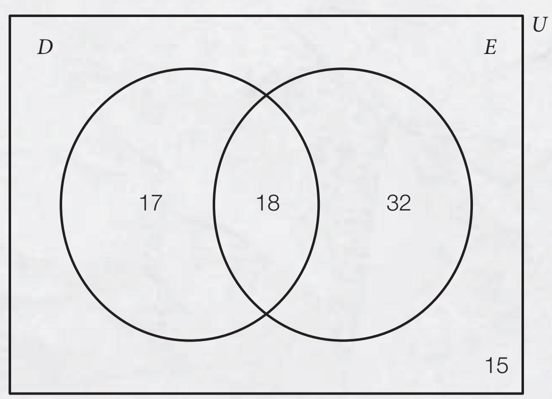
(b) If one of the football fans is selected at random, calculate the probability that the fan:

(i) supports Lazio

(ii) lives in Rome, given that the fan does not support Lazio

(iii) supports Lazio, given that the fan does not live in Rome.

8. A group of students were asked whether they owned iPhones or iPads. Their responses are illustrated on the Venn diagram.



$U = \{\text{students surveyed}\}$
$D = \{\text{students who owned iPads}\}$
$E = \{\text{students who owned iPhones}\}$

(a) How many students were surveyed?

(b) If two student are chosen at random, find the probability that:

(i) both students own an iPad

(ii) both students own iPhones, given that they own iPads

(iii) both students own iPhones, given that they do not own iPads.

9. The table below shows the number of teachers who have played any of the following types of sports in the previous week. Each teacher has played only one of the types of sports indicated.

	Male	Female	Total
Baseball		10	39
Basketball	20		
Hockey	14	19	
Total	63		110

(a) Copy and complete the table from above.

(b) If two teachers are chosen at random, calculate the probability that:

(i) only one of them has played hockey

(ii) both of them have played baseball, given that they are both female teachers

(iii) they are male teachers, given that they play hockey or basketball.

10. Seventy Economics students and teachers were asked which Economics magazine they had read in the past month. Their responses are shown in the table below. Each of the respondents had read only one of the three magazines indicated.

	The Economist	*MoneyWeek*	*New Economy*
Student	32	8	17
Teacher	6	3	4

(a) If a student is chosen at random, what is the probability that this student has read either *The Economist* or *MoneyWeek*?

(b) If two teachers are chosen at random, find the probability that at least one of them has read *New Economy*.

(c) If two of the students and teachers are chosen randomly, calculate the probability that:

(i) they are both teachers, given that they have not read *The Economist*

(ii) they are both students, given that they have either read *MoneyWeek* or *New Economy*.

11. In a group of 30 students, 20 study French, 7 study German and 6 study neither French nor German.

(a) If one student is chosen at random, calculate the probability that the student:

(i) studies both French and German

(ii) studies German, given that he/she does not study French.

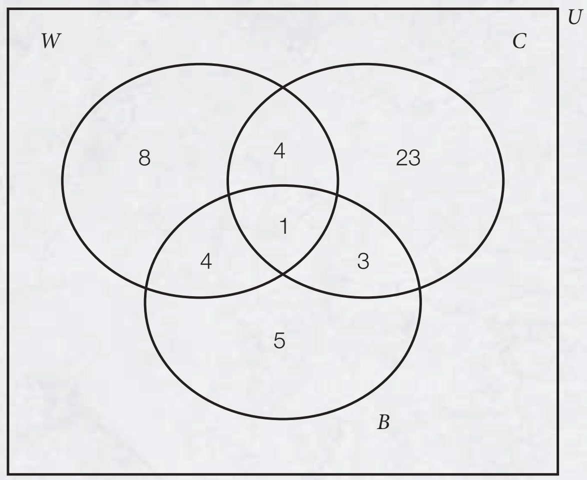
(b) Two of the students have decided to enrol on a Russian language course. Find the probability that:

(i) both students study French, given that they study either French or German but not both

(ii) both of them study German, given that they do not study French.

12. The Venn diagram below shows how 48 teachers at a Sixth Form College travelled to school in one month.

U = {48 teachers in the college}
W = {teachers who walked to school}
C = {teachers who travelled to school by car}
B = {teachers who rode to school by bike}



(a) One of the 48 teachers is asked how he/she travelled to school. Write in words what the following probabilities mean:

(i) $P(W \mid B)$

(ii) $P(C \mid B')$

(iii) $P(B' \mid W \cap C)$.

(b) Calculate all the probabilities in part (a).