6 Mathematical models

Suppose you are given a problem — it may be a practical question that a friend asks you, or an issue that you have noticed yourself to which you want to find a solution; it may even be a mathematical problem that you are set in one of your classes. How do you solve it? If it is a practical problem, you might try out different approaches or talk it through with a friend. If it is a problem encountered in class, you might go back over your notes and textbooks, looking for a similar example to see how that was solved.

The mathematician George Pólya wrote a book called *How to Solve It*, which was published in 1945. In this book he describes several distinct steps taken by people who are good at problem-solving.

These steps are:

- understand the problem
- make a plan
- carry out your plan
- look back at your solution to see whether you can improve it or use it in another context.

This process of solving problems is called 'mathematical modelling', and the stages outlined by Pólya constitute what is known as the 'modelling cycle'.

Who uses mathematical modelling? It is the normal way for mathematicians to work, as well as designers, engineers, architects, doctors and many other professionals.

If you become good at problem-solving, you can reduce the number of things that you have to memorise: it is easier to remember formulae or concepts when you understand *how* they work.

Prior learning topics

It will be easier to study this topic if you have completed:

- Chapter 2
- Chapter 14.

Chapter 17 Functions and graphs

Mathematical models allow us to use mathematics to solve practical problems, describe different aspects of the real world, test ideas, and make predictions about the future. A model may be a simplification of a real situation, but it is usually quick and cheap to work with and helps us to strengthen our understanding of the problem.

The concept of a function is fundamental to mathematical modelling. Before you can work with the more complex models in this topic, you need to be confident in your understanding and use of functions and function notation.

In this chapter you will learn:

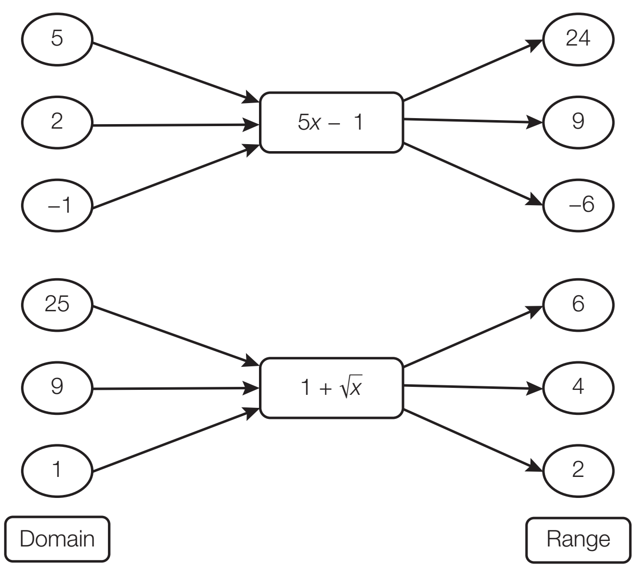
- what a function is and the definitions of domain and range
- about the graphs of some basic functions
- how to use function notation
- how to read and interpret a graph and use it to find the domain and range of a function
- about simple rational functions and asymptotes
- how to draw accurate graphs
- how to create a sketch from given information
- how to use your GDC to solve equations that involve combinations of the basic functions dealt with in this course.

Why can we use maths to describe real-world situations and make predictions? Are we imposing our own mathematical models on the world, or is it that we are discovering that the world runs according to the rules of mathematics?

17.1 What is a function?

A **function** is like a **mapping** diagram between two sets, where each element of one set maps to a single element of the other set. A function can also be viewed as a machine where you feed numbers in at one end, it works on the numbers according to a certain rule, and then gives you the output at the other end.

For example:



The set of numbers that go into the function is called the **domain**. Values in the domain are:

- the numbers chosen to describe that particular function
- values taken by the **independent** variable
- the numbers plotted on the horizontal axis when you draw a graph of that function.

You learned about dependent and independent variables in Chapter 12.

hint

Be careful not to confuse this use of the word 'range' with its use in the context of statistical data in Chapter 7.

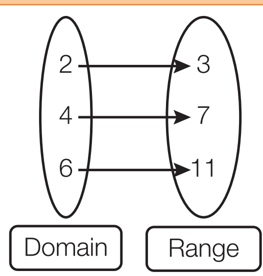
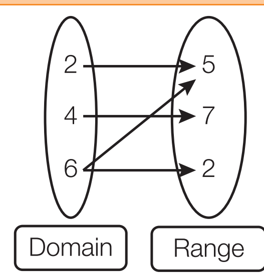
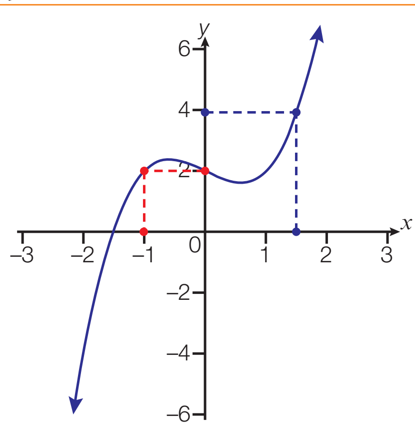
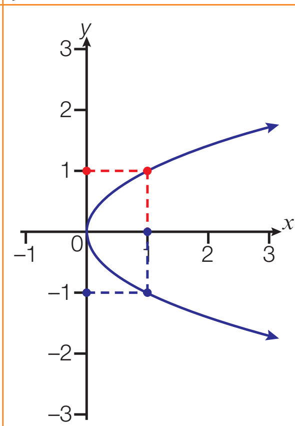
The numbers that come out of the function make up the **range** of that function. They are:

- values taken by the **dependent** variable
- the numbers plotted on the vertical axis when you draw a graph of the function.

Here are some alternative ways of thinking about a function:

- a mapping diagram
- a set of ordered pairs
- a graph
- an algebraic expression.

When representing a function by any of the above methods, you have to be careful because not all mapping diagrams, sets of ordered pairs, graphs or algebraic expressions describe a valid function. The important point to remember is that a valid function can have **only one output** for **each input value** that goes into the function.

	These are functions	These are not functions	Explanation
Mapping diagram			In the first diagram there is one output for every input. In the second diagram there are two different outputs for the input 6.
Set of ordered pairs	{(6, 8), (3, 2), (5, 9), (−1, 2)} The domain consists of the x-coordinates: {6, 3, 5, −1} The range consists of the y-coordinates: {8, 2, 9, 2}	{(6, 8) (3, 2) (6, 9) (−1, 2)} The domain consists of the x-coordinates: {6, 3, 6, −1} The range consists of the y-coordinates: {8, 2, 9, 2}	The first set of number pairs has one y-coordinate for every x-coordinate. The second set has two different y-coordinates for the x-coordinate 6.
Graph			The first graph has one y-value for every x-value. The second graph has two y-values for each positive x-value.

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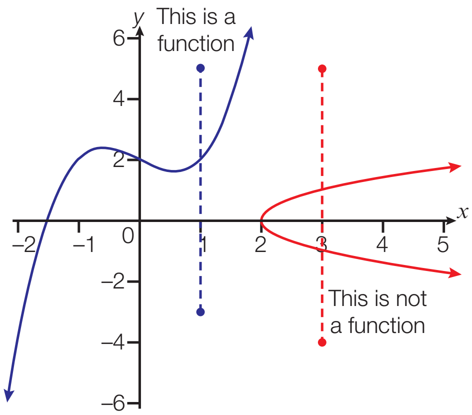
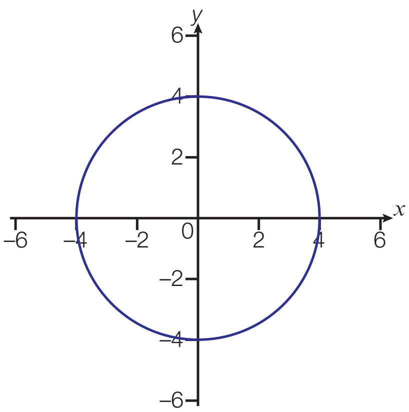
	These are functions	These are not functions	Explanation
			Graphs continued . . . This suggests the 'vertical line test': if a vertical line drawn at any x position in the domain crosses the graph only once, then the graph represents a valid function.
Algebraic expression	$f(x) = x^3 - x + 2$	$f(x) = \pm\sqrt{4 - x^2}$	The first expression will give the blue curve in the graph above. Plotting the second expression will give a circle, which fails the vertical line test.

The f(x) notation is explained further in section 17.2.

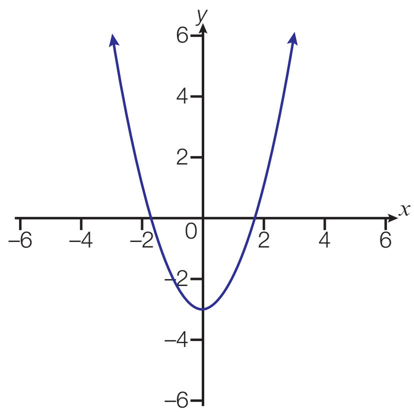
Exercise 17.1

1. For each of the graphs below, indicate whether or not the graph represents a valid function.

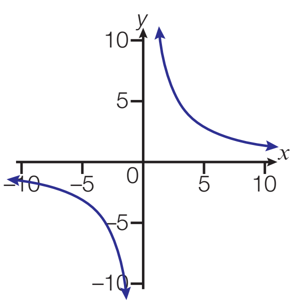
(a)



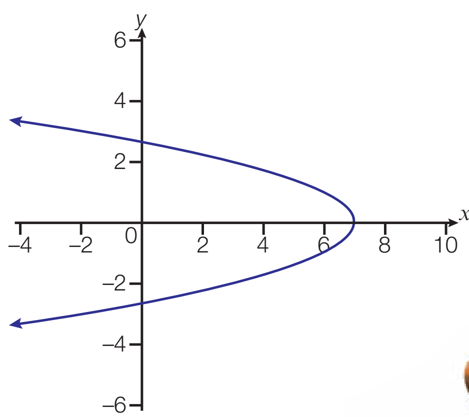
(b)



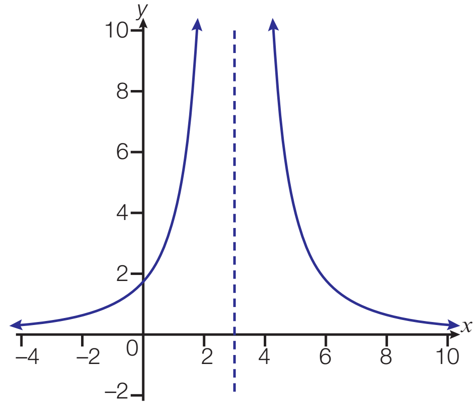
(c)



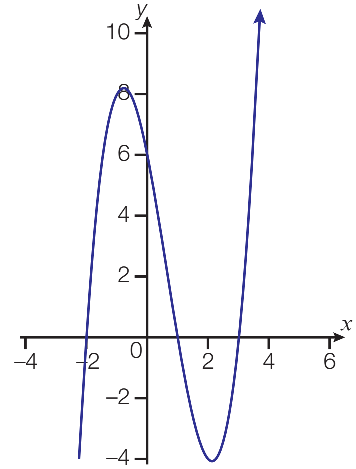
(d)



(e)



(f)



17.2 Functions in more detail

Function notation

If you are asked to draw a graph of $y = x^3 + x^2 - 5$, you know that this will be a curve because the powers of x are greater than 1, suggesting it is not linear. You can use the equation to calculate the values of y that correspond to particular values of x.

For instance, if $x = 3$, then:

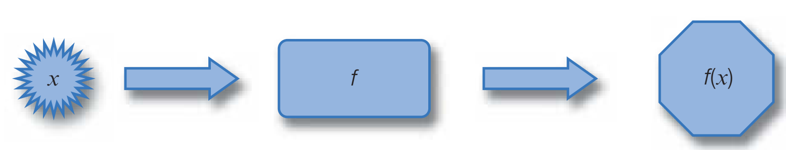
$$y = 3^3 + 3^2 - 5 = 27 + 9 - 5 = 31$$

So the point (3, 31) lies on the curve.

The mathematical relationship $y = x^3 + x^2 - 5$ can also be written in function notation $f(x) = x^3 + x^2 - 5$. The second form gives you the same information — so why do we need this different notation for expressing the same thing?

The equation $y = x^3 + x^2 - 5$ tells you the link between the x- and y-coordinates of a graph, which enables you to draw that graph.

The function notation $f(x) = x^3 + x^2 - 5$ expresses the idea of a function as a 'machine':



So x represents the input, f is the 'machine', and $f(x)$ is the output (the result of applying f to x).

The function notation $f(x)$, pronounced as 'f of x', allows us to:

- distinguish between different functions by using different letters; for example, $f(x)$, $g(x)$, $h(x)$

- use an efficient shorthand for substituting values; for example, $f(3)$ means replace x by 3 so that $f(3) = 3^3 + 3^2 - 5 = 27 + 9 - 5 = 31$
- match a formula to a physical quantity in a meaningful way; for example, we can write $h(t) = -5t^2 + 3t - 1$ to show that this is a formula to find the height at a given time.

Worked example 17.1

Q. (a) If $f(x) = 5 - 3x$, find $f(4)$, $f(-2)$ and $f(0)$.

(b) If $V(r) = \frac{4}{3}\pi r^3$, find $V(2)$, $V(12)$ and $V(21)$ in terms of π.

(c) If $f(x) = x^2 - 3x + 2$, find x when $f(x) = 42$.

(d) If $f(x) = x^3 - 2x^2 + x + 5$, find a simplified expression for $f(2x)$.

$f(4)$, $f(-2)$ and $f(0)$ mean to replace x in the expression $5 - 3x$ with 4, -2 and 0, respectively.

A. (a) $f(4) = 5 - 3 \times 4 = -7$

$f(-2) = 5 - 3 \times (-2) = 11$

$f(0) = 5 - 3 \times 0 = 5$

$V(2)$, $V(12)$ and $V(21)$ mean to replace x in the expression $\frac{4}{3}\pi r^3$ by 2, 12 and 21, respectively.

(b) $V(2) = \frac{4}{3}\pi \times 2^3 = \frac{32\pi}{3}$

$V(12) = \frac{4}{3}\pi \times 12^3 = 2304\pi$

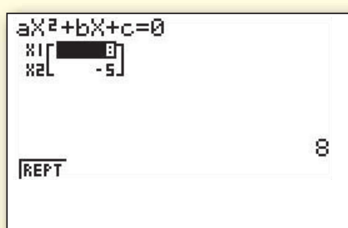
$V(21) = \frac{4}{3}\pi \times 21^3 = 12348\pi$

$f(x) = 42$ is the same as $x^2 - 3x + 2 = 42$, which is a quadratic equation that you can solve using your chosen method from Chapter 2.

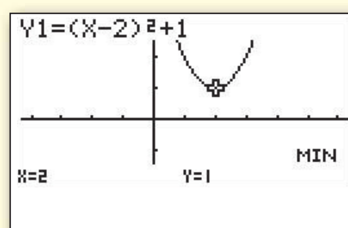
(c) $x^2 - 3x + 2 = 42$

$x^2 - 3x - 40 = 0$

TEXAS



CASIO



From GDC: $x = -5$ or $x = 8$

$f(2x)$ means to replace every occurrence of x in the expression $x^3 - 2x^2 + x + 5$ by $2x$. It is best to put brackets around $2x$, so that you will remember to cube and square the 2 as well as x.

(d) $f(2x) = (2x)^3 - 2(2x)^2 + (2x) + 5$

$= (2x) \times (2x) \times (2x) - 2 \times (2x) \times (2x) + (2x) + 5$

$= 8x^3 - 8x^2 + 2x + 5$

Exercise 17.2

1. A function is defined as $f(x) = 3x - 8$. Find the following:

(a) $f(4)$ (b) $f(-1)$ (c) $f\left(\frac{1}{3}\right)$ (d) $f(a)$

2. Given the function $g(x) = 11 - 5x$, find:

(a) $g(6)$ (b) $2g(1)$ (c) $g(7) + 13$ (d) $g(2a)$

3. Given the function $f(x) = 2x^2 - 7x + 5$, find:

(a) $f(-2)$ (b) $f(0)$ (c) $f(3) - 5$ (d) $f(c)$

4. Given that $g(x) = x^3 - 4x^2 + 3x - 7$, find:

(a) $g(-4)$ (b) $g(-1)$ (c) $13g(5)$

5. For the function $h(t) = 14t - 4.9t^2$, find:

(a) $h(1)$ (b) $h\left(\frac{1}{7}\right)$ (c) $h(3) - h(2)$ (d) $5h(10) + 200$

6. A function is defined as $f(x) = \frac{2+x}{x-2}$ for $x \neq 2$.

(a) Work out $f(-7)$.

(b) Find $f\left(\frac{3}{4}\right)$.

(c) Calculate $f(6) - f(0)$.

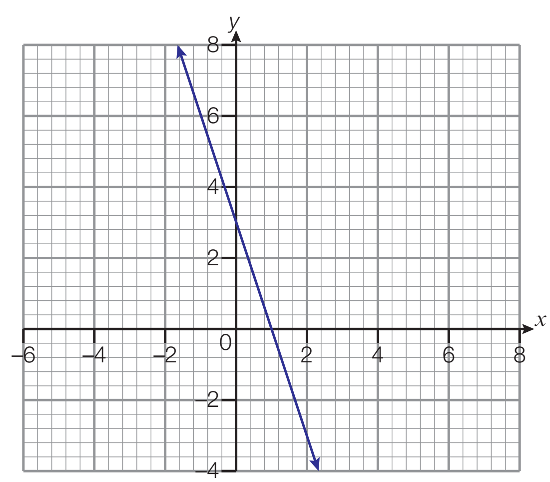
(d) Find a simplified expression for $f(x + 1)$.

Domain and range

If you are drawing a graph, the domain of a function consists of the values that are plotted along the x-axis; these are values of the independent variable. The domain may be all the real numbers on a number line, or it may be a restricted **interval** such as $-2 \leq x \leq 3$.

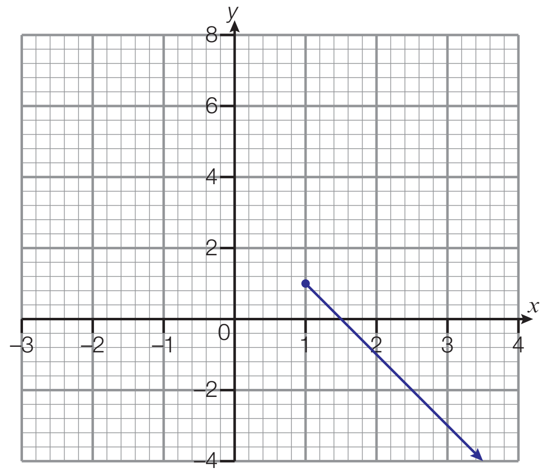
Suppose that a function is defined by $f(x) = 3 - 2x$.

If the domain is not specified, there are no values of x that cannot be used, and the graph looks like this:



Note the arrows at both ends of the line.

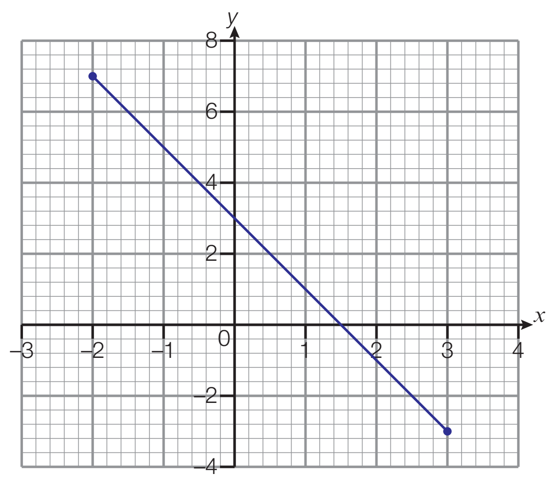
If the domain is given as $x \in \mathbb{R}, x \geq 1$, the graph looks like this:



You saw in Chapter 1 that $\mathbb{R}$ is the symbol for real numbers and in Chapter 8 that $\in$ is set notation for 'is a member of'; see Chapter 1 if you need to revise set notation for the various types of numbers and Chapter 8 if you need a reminder of set notation.

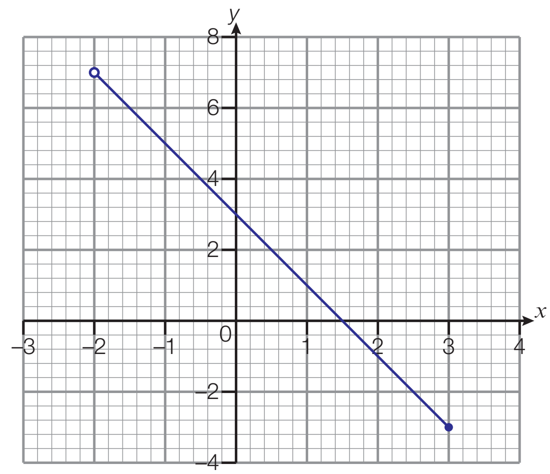
Note that there is an arrow at only one end of the line; the end at $x = 1$ is marked by a filled circle. '$x \in \mathbb{R}, x \geq 1$' indicates that the domain is all real numbers greater than or equal to 1.

If the domain is given as $-2 \leq x \leq 3$, the graph looks like this:



Now both ends, at $x = -2$ and $x = 3$, are marked by filled circles.

If the x value of one of the ends is not included in the domain, that end would be marked with an empty (non-filled) circle. For example, if the domain were $-2 < x \leq 3$ (which means $x = -2$ is not included in the domain), then the graph would look like this:



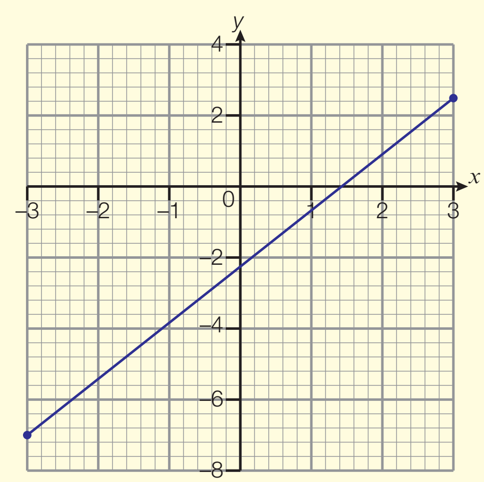
The range also changes depending on the domain:

- For $f(x) = 3 - 2x$, $x \in \mathbb{R}$, the range is also $f(x) \in \mathbb{R}$.
- For $f(x) = 3 - 2x$, $x \in \mathbb{R}$, $x \geq 1$, the range is $f(x) \leq 1$, as you can see from the first graph on page 499.
- For $f(x) = 3 - 2x$, $x \in \mathbb{R}$, $-2 \leq x \leq 3$, the range is $-3 \leq f(x) \leq 7$, as you can see from the second graph on page 499.

Worked example 17.2

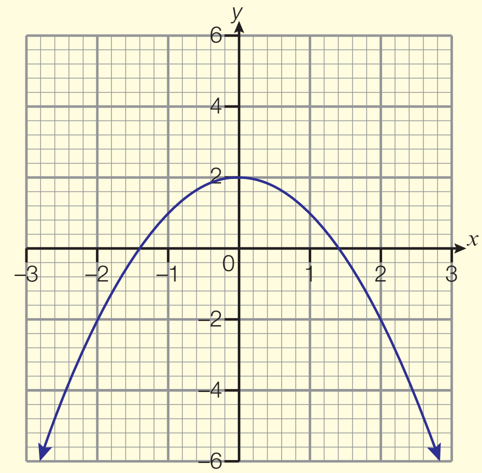
Q. Write down the domain and range for each of these functions:

(a)

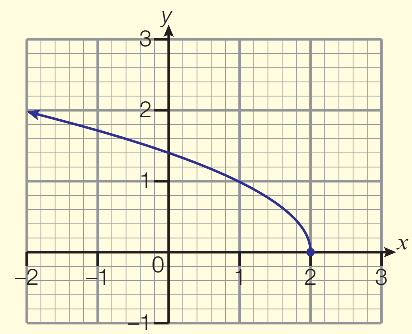


continued . . .

(b)



(c)



The graph has filled circles at both ends, so the domain is restricted between, and also includes, the x-coordinates of these ends. The range is between the y-coordinates of these ends.

A. (a) The domain is $-3 \leq x \leq 3$.

The range is $-7 \leq f(x) \leq 2.5$.

The graph has arrows at both ends, so the domain is all real numbers. The curve has a maximum point at (0, 2), so the range is all numbers less than or equal to 2.

(b) The domain is $x \in \mathbb{R}$.

The range is $f(x) \leq 2$.

There is an arrow at one end of the curve, which means that the domain (and range) is unrestricted in this direction; a filled circle at the other end means that the domain reaches up to $x = 2$ but no further. The minimum y-value on the curve is 0.

(c) The domain is $x \leq 2$.

The range is $f(x) \geq 0$.

If you use your GDC to draw the graph of a function, you can find the coordinates of important points on the curve, such as minimum and maximum points or the points at the beginning and end of the domain. See section '*22.2G(e)*' on page 648 or sections '*22.3.17*' and '*22.3.18*' on pages 678–683 of the GDC chapter for a reminder of the different methods.

Worked example 17.3

Q. Use your GDC to draw graphs of the following functions. State the domain and range in each case.

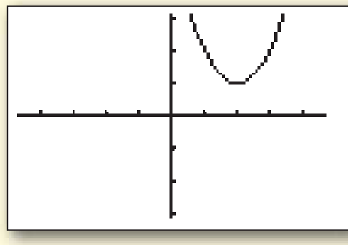
(a) $f(x) = (x - 2)^2 + 1$

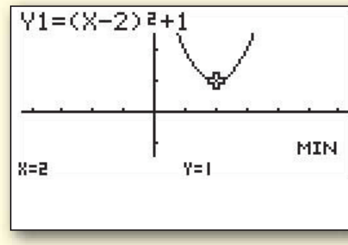
(b) $f(x) = x^3 - x;\ -2 \leq x \leq 2$

(c) $f(x) = 2^{x-1}$

Plot the graph on your GDC and find the coordinates of the minimum point on the parabola; this gives you the lowest value of the range. There is no restriction given on the domain, so it consists of all real numbers.

A. (a) TEXAS CASIO





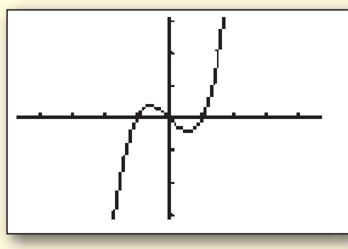
The minimum point on the parabola is at (2, 1)

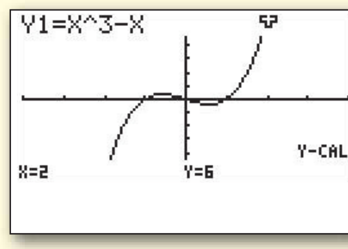
The domain is $x \in \mathbb{R}$.

The range is $f(x) \geq 1$.

In this case the domain is restricted to x values between, and including, –2 and 2. Use your GDC to find the coordinates of the points at the beginning and end of the domain.

(b) TEXAS CASIO



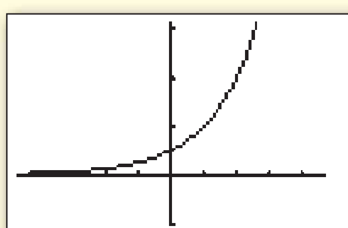


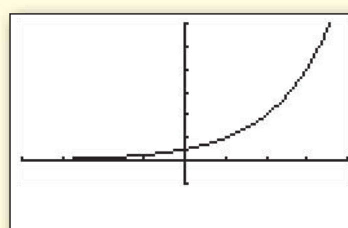
The domain is $-2 \leq x \leq 2$.

The range is $-6 \leq f(x) \leq 6$.

The domain can take all real values, but the graph tells you that values in the range are always above zero.

(c) TEXAS CASIO





The domain is $x \in \mathbb{R}$.

The range is $f(x) > 0$.

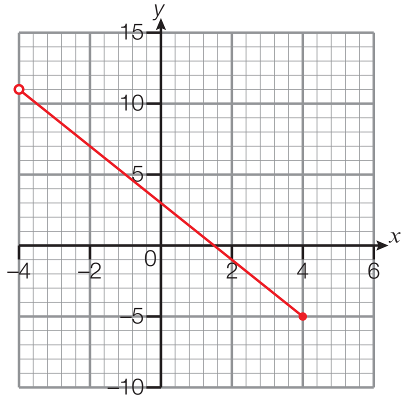
The curve gets close to, but never touches, the x-axis, so there is a horizontal asymptote at $y = 0$.

FF You will learn about horizontal and vertical asymptotes in section 17.3.

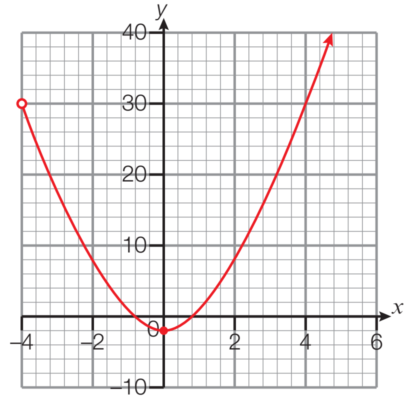
Exercise 17.3

1. Write down the domain and the range for each of the following functions.

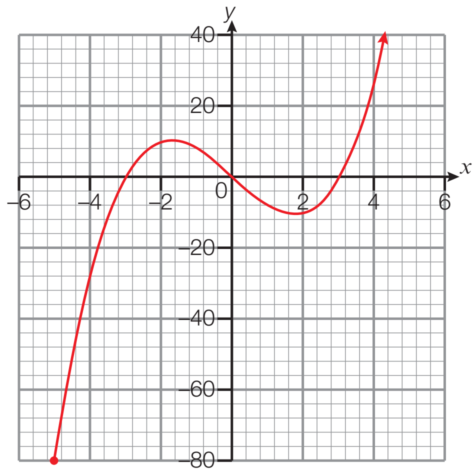
(a)



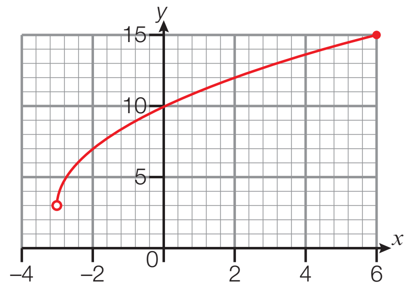
(b)



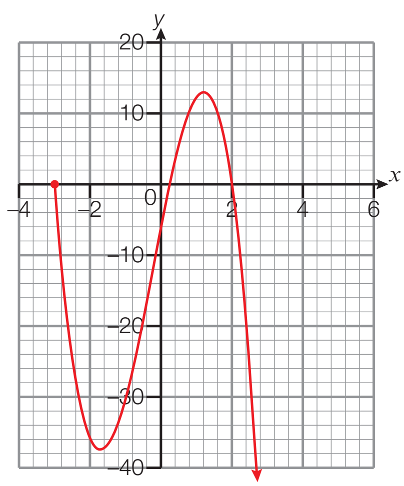
(c)



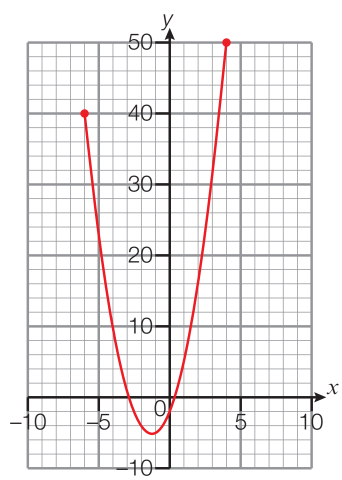
(d)



(e)



(f)



2. Use your GDC to draw the graphs of the following functions. State the range of each function.

   (a) $y = 4x - 7; -1 \le x \le 12$

   (b) $y = x^2 + 3; -7 \le x < 7$

   (c) $y = x^3 - 4x; -5 \le x \le 4$

   (d) $y = (x + 5)(2x - 1); -6 < x < 6$

17.3 Rational functions

hint

Compare this with the term 'rational number' introduced in Chapter 1.

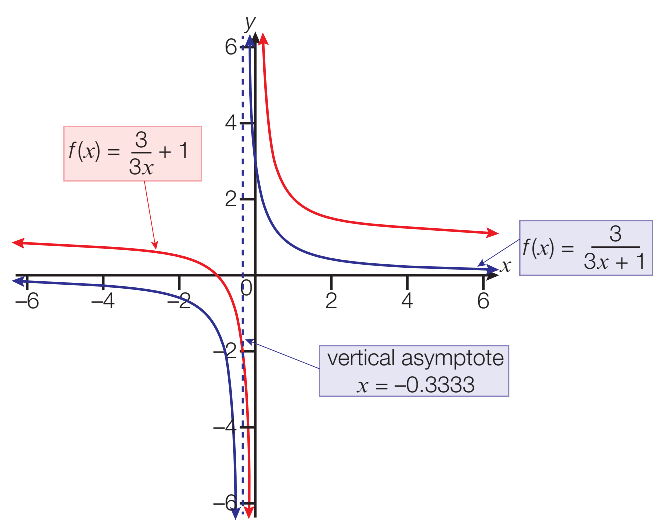
A rational function is a function whose algebraic expression looks like a fraction or ratio.

The simplest form of a rational function is $f(x) = \dfrac{a}{bx + c}$ where a, b and c are constants.

The graph of such a function has a distinctive shape, known as a **hyperbola**. In fact, the graph is made up of two curves; the values of a, b and c determine the positions of these curves.

By using a GDC or a maths software package on the computer, you can investigate how the constants a, b and c affect the basic shape of the graph. Be careful entering the function's formula: make sure to use brackets and division signs correctly.

For example, to enter $f(x) = \dfrac{3}{3x+1}$, you need brackets to tell the GDC that it is dividing 3 by all of $(3x + 1)$. If you did not have the brackets around $3x + 1$, the GDC may draw the graph of $f(x) = \dfrac{3}{3x} + 1$, which is a different function, as you can see in the graph below.



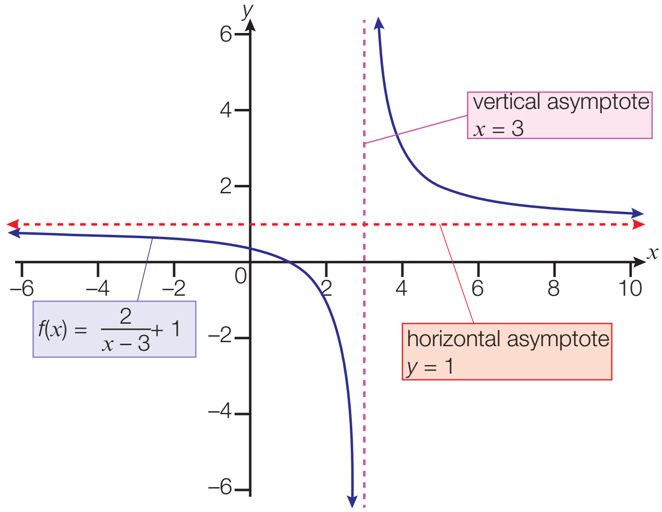
Try exploring hyperbola graphs yourself. Here are some suggestions:

- Start with the simplest rational function $f(x) = \dfrac{a}{x}$ and draw graphs for different values of a.

- Draw graphs of $f(x) = \dfrac{1}{x \pm c}$ for different positive numbers c.
- Draw graphs of $f(x) = \dfrac{1}{x \pm c} + d$ for various values of d.

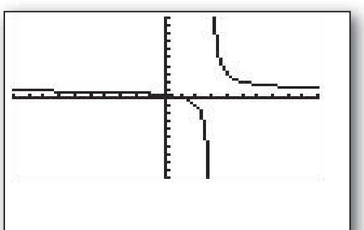
You will find that each graph has both a vertical and a horizontal 'break'.

For example, the graph of $f(x) = \dfrac{2}{x-3} + 1$ looks like:

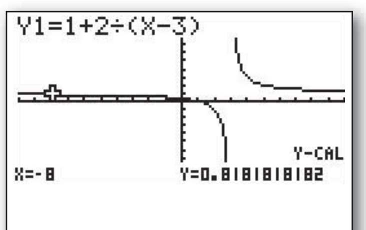


With your GDC, you can use a trace function or the table of coordinates to find the approximate location of the 'breaks' in the graph, which are called **asymptotes**. See sections '*17.2 Finding a vertical asymptote*' and '*17.3 Finding a horizontal asymptote*' on pages 679 and 680 of the GDC chapter if you need to.

TEXAS



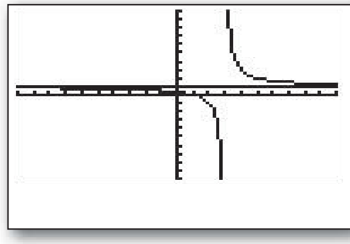
CASIO



The horizontal break in the graph is called a **horizontal asymptote**. It occurs whenever a function's output (y) value approaches a certain number as x increases or decreases.

In the case of $f(x) = \dfrac{2}{x-3} + 1$, we can see that as x increases, $f(x)$ approaches 1. The fraction $\dfrac{2}{x-3}$ gets smaller and smaller because the denominator is getting larger and larger, so $f(x)$ approaches 1 but will never actually reach 1. Therefore, the equation of the horizontal asymptote is $y = 1$. You can draw $y = 1$ on your GDC to confirm that this is the location of the break.

TEXAS



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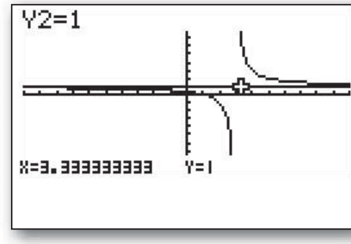


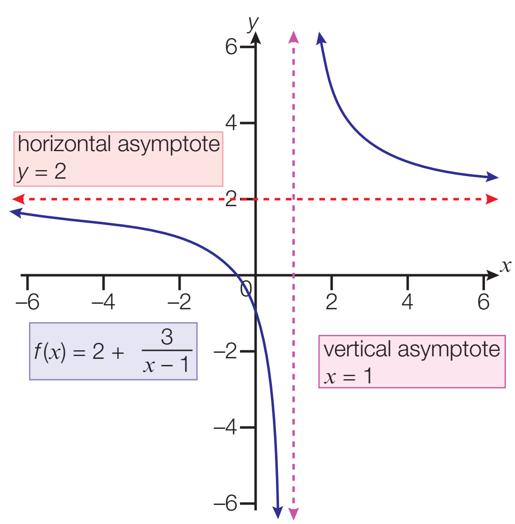
The vertical break in the graph is called a **vertical asymptote**. It occurs when a fraction's denominator becomes zero.

In general, the graph of $f(x) = \frac{a}{x \pm c}$ will have a vertical asymptote when $x \pm c = 0$. So:

- the graph of $f(x) = \frac{2}{x+3}$ will have a vertical asymptote when $x + 3 = 0$, i.e. at $x = -3$
- the graph of $g(x) = \frac{1}{2x-5}$ will have a vertical asymptote when $2x - 5 = 0$, i.e. at $x = 2.5$.

In both of the above cases, the horizontal asymptote is the line $y = 0$ (in other words, the x-axis).

To draw the graph of $h(x) = 2 + \frac{3}{x-1}$, note that there will be a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.

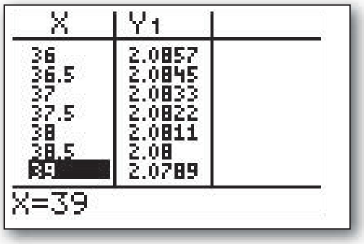


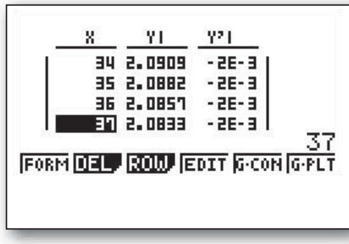
Asymptotes can sometimes be difficult to identify on the GDC screen. If you're having trouble locating the asymptotes on your GDC graph, check the 'table' function for the list of coordinates of the points on the graph. (See '*14.1 Accessing the table of coordinates from a plotted graph*' on page 678 of the GDC chapter if you need to.)

When identifying an asymptote from a table of coordinates:

- a horizontal asymptote is found by looking at what happens to the y values as the x values get larger:

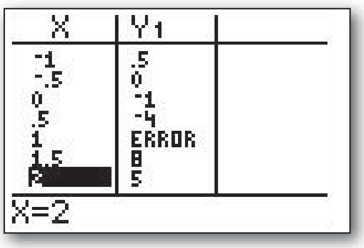
TEXAS **CASIO**

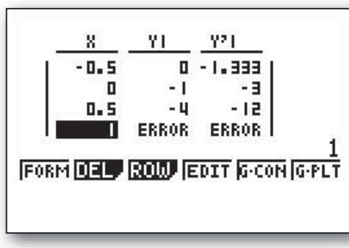




- a vertical asymptote occurs when the table shows an ERROR message:

TEXAS **CASIO**





Worked example 17.4

Use your GDC to sketch the graph. (See '*22.2G Graphs*' on page 645 of the GDC chapter if you need to.)

Q. A function is defined by $g(x) = 3 - \dfrac{5}{x+2}$ for $-6 \le x \le 3$.

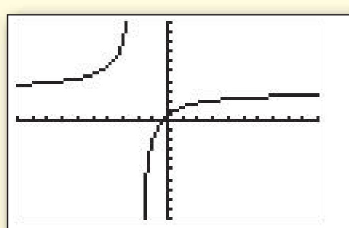
(a) Sketch its graph.

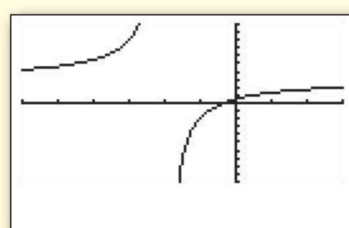
(b) Write down the equation of the vertical asymptote.

(c) What is the equation of the horizontal asymptote?

A. (a)

TEXAS **CASIO**





The denominator of the fraction equals zero when $x + 2 = 0$.

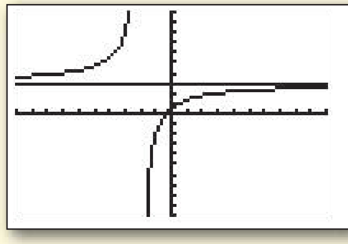
From the GDC graph, it looks as though the horizontal asymptote is at $y = 3$. Draw the line $y = 3$ on your GDC to check this, or use the table of coordinates to look at the y values as x increases. See '*22.2G*' and/or '*17.3*' of the GDC chapters as appropriate.

continued . . .

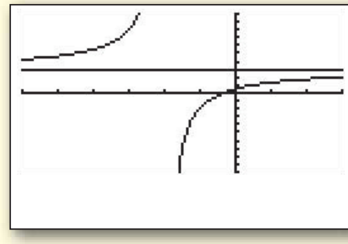
(b) The vertical asymptote is $x = -2$.

(c)

TEXAS



CASIO



The horizontal asymptote is $y = 3$.

Exercise 17.4

1. By drawing the graphs of the following functions, in each case:

(i) state the equation of the vertical asymptote

(ii) find the horizontal asymptote.

(a) $y = \frac{1}{x}$ (b) $y = \frac{3}{x}$ (c) $y = \frac{1}{x+1}$ (d) $y = \frac{2}{x+1}$

(e) $y = \frac{1}{2x+1}$ (f) $y = \frac{4}{2x-3}$ (g) $y = 5 - \frac{7}{2x-3}$ (h) $y = \frac{1}{3x-2} + 4$

17.4 Drawing graphs and diagrams

As you have seen, graphs and diagrams are very powerful tools. They can help you to visualise and discover properties of the situation you are studying. Before moving on to mathematical models in the next few chapters, you need to:

- understand the difference between a sketch and an accurate graph
- be able to create a sketch from information that you have been given
- be able to transfer a graph from your GDC onto paper
- know how to draw an accurate graph
- be able to draw combinations of two or more graphs on one diagram
- know how to use your graphs to solve equations that involve combinations of functions that you have studied.

The difference between a sketch and an accurate graph

It is important to understand the difference between the instructions 'plot', 'sketch' and 'draw', and be able to choose the appropriate type of graph for a particular problem. Read any problem carefully and decide which kind of diagram will be the most useful initially; you can always draw another if the first does not give you sufficient clarity. It is also important to make sure that any diagram you draw is large enough! A tiny diagram, squashed into one corner of the paper, will not help you; nor will it be informative to the person who is checking or marking your work!

Sketch	A sketch represents a situation by means of a diagram or graph. It should be clearly labelled and give a general idea of the shape or relationship being described. All obvious and relevant features should be included, such as **turning points**, **intercepts** and **asymptotes**. The dimensions or points do not have to be drawn to an accurate scale/position.
Plot	To plot a graph, you need to calculate a set of accurate points and then mark them clearly, and accurately, on your diagram using a correct scale.
Draw	A drawing is a clearly labelled, accurate diagram or graph. It should be drawn to scale. Straight lines should be drawn with a ruler, and points that are known not to lie in a straight line should be joined by a smooth curve.

When you are drawing, plotting or sketching:

- Use a pencil — you may need to make changes, and using permanent ink would make this difficult.
- Mark the axes for the graph— and remember to label them. Not all graphs are x–y graphs; for example, you might be plotting height against time. Always use the correct labels.
- Follow any instructions relating to the scale, the domain or the range of a graph.

Does a graph without labels or scales still have meaning?

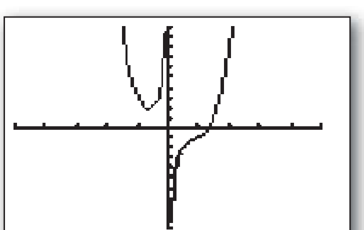
Creating a sketch

Draw the function $f(x) = x^4 - x^2 - \dfrac{1}{x}$ on your GDC. What do you notice?

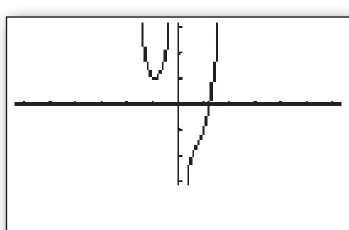
What are the most important features to include in a sketch?

You should have obtained an image like this:

TEXAS



CASIO

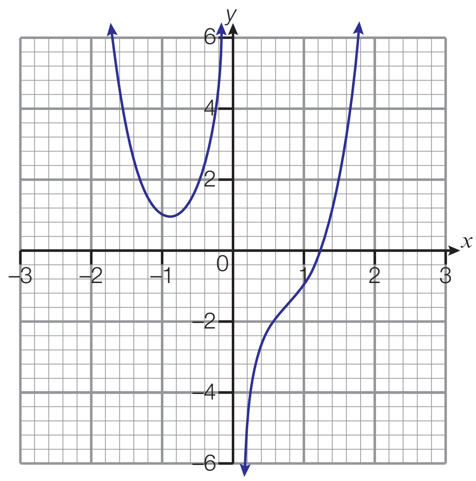


The points to notice are:

- This is an x–y graph, so the axes should be labelled x and y.
- The curve on the left of the y-axis has a minimum at approximately $(-1, 1)$.
- There is a vertical asymptote at $x = 0$ (the y-axis).
- There is an x-intercept between the points $(1, 0)$ and $(2, 0)$.
- There is no y-intercept.

Did you observe these features on your GDC graph?

Now you can make a sketch of the graph, by transferring these important features onto paper.



Draw the graph on your GDC. The curve is decreasing, so the value of the function at $x = 0$ is the maximum of the range, and the value at $x = 6$ is the minimum of the range; these values can be found with your GDC. See section '*22.2G(e) The trace function*' on page 648 of the GDC chapter if you need to.

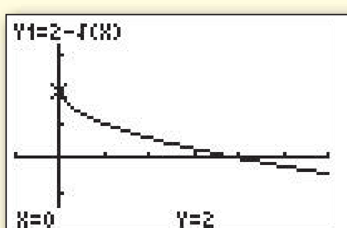
Worked example 17.5

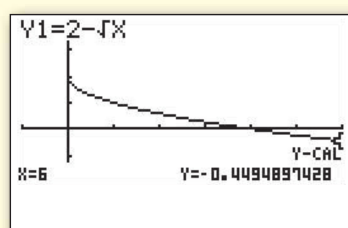
Q. (a) Sketch the graph of $f(x) = 2 - \sqrt{x}$ for the domain $0 \le x \le 6$. Comment on the range of $f(x)$.

(b) Sketch the graph of $h(x) = \sqrt{2 - x}$ for the same domain as in (a). What do you notice about the range of this function?

A. (a)

TEXAS | **CASIO**





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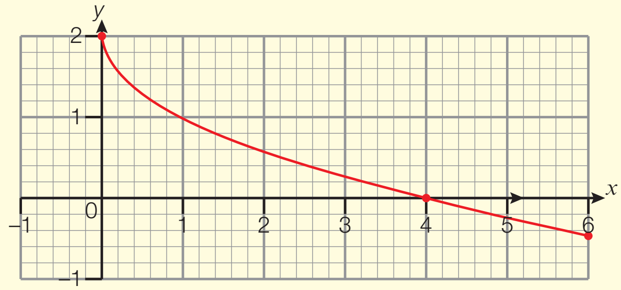
Transfer the sketch from your GDC onto paper; make sure you indicate the domain using filled circles at the ends of the curve.

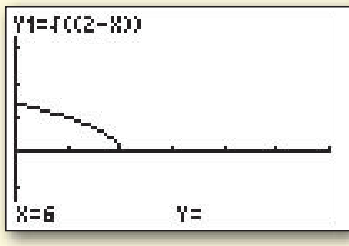


The range is $-0.449 \leq f(x) \leq 2$.

(b)

Draw the graph on your GDC. This graph has no values outside $0 \leq x \leq 2$, and it is decreasing over this interval, so the maximum and minimum values are the y values at $x = 0$ and $x = 2$.

TEXAS



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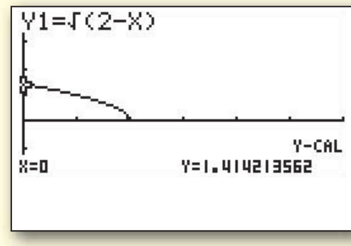
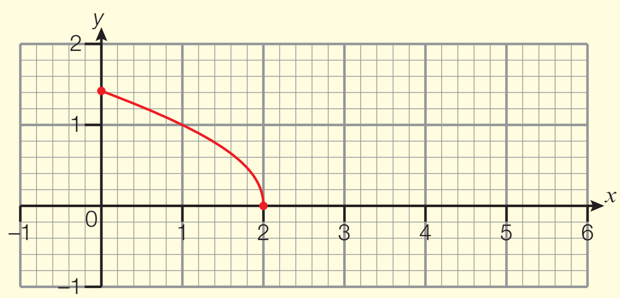


Transfer the sketch from your GDC onto paper. Even though the graph has no values outside $0 \leq x \leq 2$, you should still make the x-axis extend to $x = 6$, so that it includes all of the given domain.



The range is $0 \leq f(x) \leq 1.41$.

When x gets bigger than 2, the quantity inside the square root becomes negative. It is impossible to find the square root of a negative number, so this has restricted the domain and range of (x).

Plotting a graph

To **plot** a graph, you need to calculate the points that you are going to put on the graph. You can do this by substituting x values into the equation by hand or by using the table facility on your GDC. (See *'14.1 Accessing the table of coordinates from a plotted graph'* if you need to.)

Worked example 17.6

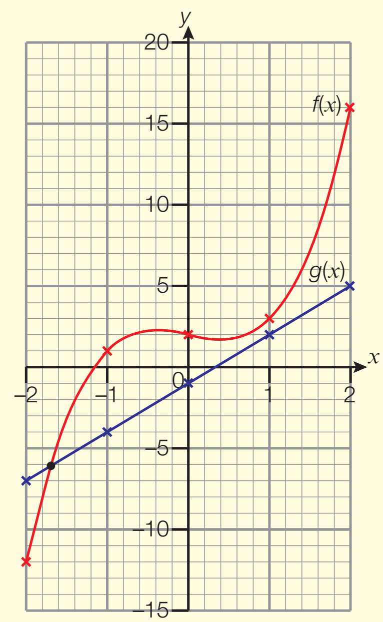
Q. If $f(x) = 2x^3 - x + 2$ and $g(x) = 3x - 1$, plot $f(x)$ and $g(x)$ on the same graph. Use your graph to find the coordinates of the point where the two functions intersect. Take the domain to be $-2 \leq x \leq 2$.

A.

x	−2	−1	0	1	2
$f(x) = 2x^3 - x + 2$	−12	1	2	3	16

x	−2	−1	0	1	2
$g(x) = 3x - 1$	−7	−4	−1	2	5

Plot the points for each function. For $f(x)$, draw a curve through the plotted points; for $g(x)$, draw a line through the plotted points.



You can see from the graph that there is only one intersection. Estimate the coordinates of this intersection point.

The two functions cross at the point (−1.7, −6.1).

exam tip

When the solution is taken from a hand-drawn graph, as in Worked example 17.6, it is difficult to be accurate to more than one decimal place. As long as your graph is clearly drawn with the intersection point marked, you will be given credit for an answer within certain tolerances, e.g. (−1.7 ± 0.2, −6.1 ± 0.2).

Exercise 17.5

1. Consider the function $f(x) = \sqrt{1 - x}$.

(a) Sketch $f(x)$ for the domain $-4 \leq x \leq 4$.

(b) Comment on the range of $f(x)$.

(c) Let $g(x) = 5 + f(x)$ for $-4 \leq x \leq 4$. Deduce the range of $g(x)$. Justify your answer.

2. (a) Sketch the graph of the function $f(x) = 3^x$ for the domain $-1 \leq x \leq 3$, and write down the range of $f(x)$.

(b) Sketch the graph of the function $g(x) = 3^x + \left(\frac{1}{3}\right)^x$ for the domain $-1 \leq x \leq 3$, and state the range of $g(x)$.

(c) Sketch the graph of the function $h(x) = 3^x - \left(\frac{1}{3}\right)^x$ for the same domain, and state the range of $h(x)$.

(d) Using your results from above, deduce the range of $g(x) + h(x)$. Justify your answer.

3. For each of the following functions, use your GDC to draw the graph, then look at the main features of the graph and hence sketch it on paper. Take the domain of each function to be $-3 \leq x \leq 3$.

(a) $f(x) = \dfrac{5x+1}{2x-3}$

(b) $f(x) = \dfrac{x+7}{x^2-4} + 5$

(c) $f(x) = 2 - \dfrac{x^2}{x+1}$

(d) $f(x) = x^2 + x - \dfrac{8}{x^2}$

4. Plot the graphs of the following pairs of functions for the domain $-3 \leq x \leq 3$. In each case give the coordinates of the points where the two functions intersect to 2 s.f.

(a) $f(x) = x^2 + x - \dfrac{1}{x}$, $g(x) = 2x - 1$

(b) $f(x) = 2^x - x^2$, $g(x) = 2x - 1$

(c) $f(x) = 3 \times 2^x + 1$, $g(x) = 1 - x$

(d) $f(x) = \dfrac{5x+1}{2x-1}$, $g(x) = x + 2$

Solving equations using graphs on your GDC

As you saw in Chapter 2, there are standard algebraic methods for solving linear, simultaneous and quadratic equations. However, many equations are very difficult to solve with algebra, and can often be solved much more easily by looking at a graph. Using a graph is also helpful because, as long as the domain and range are well chosen, it can show you **all** the solutions to an equation. If you use an equation solver on your GDC, you might find only one solution when there are several.

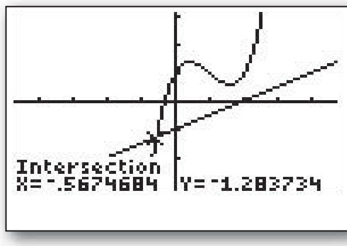
To solve equations with a graph, draw the function on the left-hand side of the equals sign as one line or curve, and the function on the right-hand side of the equation as another line or curve on the same graph; then look at where they intersect. The x-coordinate of each intersection will be a root of the equation.

For example, to solve the equation $x^3 - 3x^2 + 2x + 1 = \frac{1}{2}x - 1$, take the two functions $f(x) = x^3 - 3x^2 + 2x + 1$ and $g(x) = \frac{1}{2}x - 1$ and draw their graphs on the same axes.

The graph of $g(x)$ is a straight line, while the graph of $f(x)$ is a cubic curve with two turning points. You might expect that the equation will have three solutions, but by drawing the actual graphs on the same set of axes you will see that there is only one intersection.

You can use the intersection tool on your GDC to locate the point of intersection (see '*19.2 (a) Solving unfamiliar equations using a graph*' on page 684 of the GDC chapter for a reminder of how to use this tool if you need to).

TEXAS



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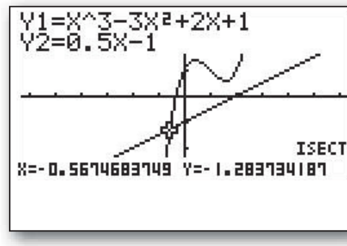


GDC

Remember that you should round the answers given by the GDC to three significant figures if no other specific degree of accuracy has been requested.

The GDC tells you that the curves intersect at $(-0.567, -1.28)$, so the solution to the equation $x^3 - 3x^2 + 2x + 1 = \frac{1}{2}x - 1$ is $x = -0.567$.

You can learn and practise this technique by plotting graphs on paper too.

Worked example 17.7

Q. The functions $f(x)$ and $h(x)$ are defined as $f(x) = 2^x + 1$ and $h(x) = 9 - x^2$ over the domain $-4 \le x \le 4$.

(a) Fill in the missing values in the following table.

x	-4	-3	-2	0	1	3	4
$f(x)$	1.0625		1.25	2		9	
$h(x)$	-7	0		9	8		-7

(b) Using a scale of 2 cm to represent one unit on the x-axis and 1 cm to represent two units on the y-axis, draw both functions on the same graph.

(c) Use your graph to find the solutions to the equation $2^x + 1 = 9 - x^2$. Give the answer to 1 d.p.

Substitute the given values of x into the algebraic expressions for $f(x)$ and $h(x)$. Label each curve.

Plot the points from (a) on a set of axes drawn according to the given scale. Then draw a curve through each set of points. Label each curve.

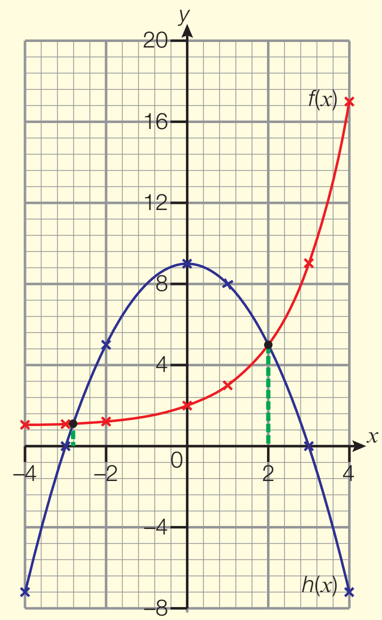
Read off the x-coordinates of the two points of intersection. It can be helpful to draw dashed lines from the points of intersection to the x-axis.

continued . . .

A. (a)

x	−4	−3	−2	0	1	3	4
$f(x)$	1.0625	1.125	1.25	2	3	9	17
$h(x)$	−7	0	5	9	8	0	−7

(b)



(c) The solutions are $x = 2.0$ and $x = -2.8$.

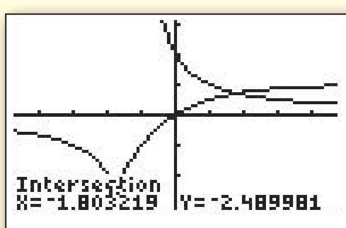
Worked example 17.8

Q. Find the solutions of the equation $\dfrac{2}{x+1} = 1 - 2^{-x}$.

Plot the graphs of $y = \frac{2}{x+1}$ and $y = 1 - 2^{-x}$ on your GDC. Be careful when you enter the equations into your calculator. Use brackets for $(x + 1)$ and $2^{(-x)}$ to make sure that your GDC does the intended calculations.

A.

TEXAS



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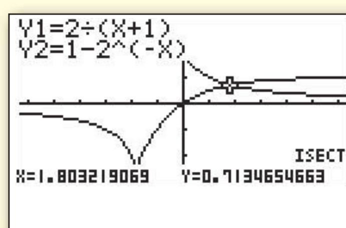


Find the x-coordinates of the two points of intersection. (See '*19.2 (a) Solving unfamiliar equations using a graph*' on page 684 of the GDC chapter for a reminder of how to use this tool if you need to).

The solutions are $x = -1.80$ and $x = 1.80$.

Exercise 17.6

1. The functions $f(x)$ and $g(x)$ are defined as $f(x) = x^2 - 2x$ and $g(x) = 2 - x$ for $-2 \leq x \leq 4$.

(a) Copy the following table and fill in the missing values.

x	-2	-1	0	1	2	3	4
$f(x)$			0				8
$g(x)$			2				-2

(b) Draw both functions on the same set of axes.

(c) Use your graph to find the solutions to the equation $x^2 - 2x = 2 - x$.

2. The functions $f(x)$ and $g(x)$ are defined as $f(x) = 2x^3 - 1$ and $g(x) = 5x$ for $-2 \leq x \leq 2$.

(a) Copy the following table and fill in the missing values.

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					

(b) Draw both functions on the same set of axes on graph paper.

(c) Use your graph from (b) to find the solutions to the equation $2x^3 - 1 = 5x$. Give your answer to 1 d.p.

3. The functions $f(x)$ and $g(x)$ are defined as $f(x) = 3^x - 2$ and $g(x) = 1 - 2x - x^2$ for $-4 \leq x \leq 2$.

(a) Copy the following table and fill in the missing values.

x	-4	-3	-2	-1	0	1	2
$f(x)$			-1.89				7
$g(x)$			1				-7

(b) Draw both functions on the same set of axes on graph paper.

(c) Use your graph to find the solutions to the equation $3^x - 2 = 1 - 2x - x^2$. If your answer is not exact, give it to 1 d.p.

4. Let $f(x) = x^3 - 3x^2 + 5x + 2$ and $g(x) = 7x - 5$.

(a) Use your GDC to draw both functions on the same set of axes.

(b) State the number of intersections of the two functions.

(c) Hence find all the solutions of the equation $f(x) = g(x)$ to 1 d.p.

5. For each of the following pairs of functions, draw both functions on the same set of axes on your GDC, and hence:

(i) state the number of intersections of the two functions

(ii) find all the solutions to the equation $f(x) = g(x)$.

(a) $f(x) = \sqrt{5x+7}$ and $g(x) = x^3 + 2.451$

(b) $f(x) = 2^{x+1}$ and $g(x) = 3^x$

(c) $f(x) = 6.15 \times 9^x$ and $g(x) = 2.78 \times 10^{2x}$

(d) $f(x) = 2^x - 4$ and $g(x) = \dfrac{3x}{x-2}$

(e) $f(x) = x^3 - 7x + 1$ and $g(x) = 8 - 5^x$

(f) $f(x) = 3 - x^2$ and $g(x) = \dfrac{x-1}{7-4x}$

6. Solve the following equations with your GDC, using the graphical method.

(a) $x^2 = 3x + 6$

(b) $-3x^3 + 2 = 8x^2 - 3x$

(c) $1.5 - x^3 = 4x^2 + x - 3.142$

(d) $x^4 - x^2 - 9 = x^2 + 6$

(e) $x^4 + 5x^2 + 0.2 = 4x^3 + 3x - 1$

(f) $1.7x^2 + 0.875x - 1.4 = 1.5x^4 + 4x^3 - 0.5$

(g) $(x - 2)(x + 1)(x + 4) = 0.732x + 1.926$

7. Solve the following equations with your GDC, using the graphical method.

(a) $1.02^t = 5$

(b) $40^{0.0125x} = 25^{0.336x}$

(c) $17 \times 2^{0.47t} = 23$

(d) $28^{(5x + 0.336)} = 43^{0.127x}$

(e) $2^x + 5^x = 7^x + 0.125$

(f) $3.958^x = x^3 - 7x - 4$

(g) $2.2 - \left(\dfrac{3}{4}\right)^x = 9.645 \times 10^x$

(h) $\left(\dfrac{1}{3}\right)^x = 2.632 - 5^{2-x}$

Summary

You should know:

- the concept of a function and the definitions of domain and range
- how to recognise and use function notation
- how to interpret the graph of a function and use it to find the domain and range
- how to recognise and identify simple rational functions and asymptotes
- how to draw or plot accurate graphs
- how to sketch a graph from information given
- how to use a GDC and the hand-drawn graphical method to solve equations that involve combinations of the functions studied in this course.

Mixed examination practice

Exam-style questions

1. A function is defined as $f(x) = 7x - 13$.

(a) Find $f(0)$.

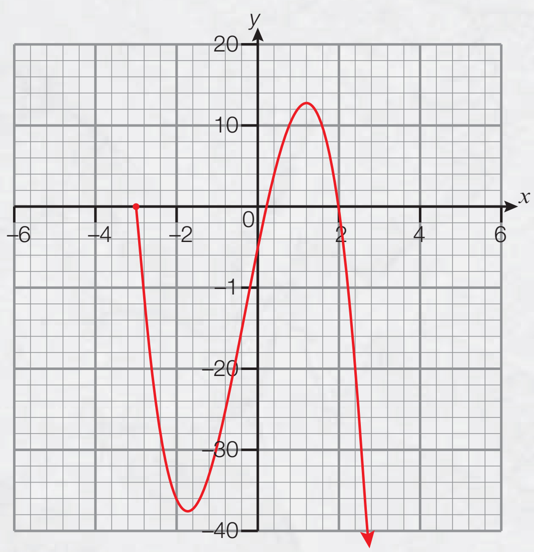
(b) State the value of $f(3)$.

(c) Find an expression for:

(i) $f(a)$

(ii) $f(a-2)$

2. The graph of the function $f(x)$ is shown below.



(a) Write down the domain of the function.

(b) State the range of $f(x)$.

(c) Given that $f(k) = 0$, find all the possible values of k.

3. Sketch the graphs of the following functions for $-5 \le x \le 5$. In each case:

(i) State the equation of the vertical asymptote.

(ii) Find the horizontal asymptote.

(a) $f(x) = \dfrac{3}{2-x}$

(b) $g(x) = 4 + \dfrac{5}{9x-7}$

4. Use your GDC to draw the graph of the function $f(x) = \frac{7x-5}{1-8x} + 4$, with domain $-3 \le x \le 3$.
Look at the main features of the graph and hence sketch the function on paper.

5. The functions $f(x)$ and $g(x)$ are defined as $f(x) = 1 - 4x - x^3$ and $g(x) = 7 - 2x$ for $-3 \le x \le 3$.

   (a) Copy the following table and fill in the missing values.

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

   (b) Draw both functions on the same set of axes.

   (c) Use your graph from (b) to find the solutions to the equation $1 - 4x - x^3 = 7 - 2x$. Give your answer to 1 d.p.

6. Two functions are defined as $f(x) = 3 + 5^{x-1}$ and $g(x) = 4 + 3x + 6x^2 - x^4$. Both functions have the same domain, $-3 \le x \le 3$.

   (a) Use your GDC to draw both functions on the same set of axes.

   (b) State the number of intersections of the two functions.

   (c) Hence find all the solutions of the equation $f(x) = g(x)$ to 1 d.p.

Chapter 18 Linear and quadratic models

In the introduction to Topic 6, you saw an outline of the steps of **mathematical modelling** and problem-solving described by the mathematician George Pólya:

1. understand the problem
2. make a plan
3. carry out your plan
4. look back at your solution to see whether you can improve it or use it in another context.

Understanding the problem can take time. You need to clarify what you need to find or show, draw diagrams, and check that you have enough information. It may even be that you have too much information and need to decide what is relevant and what is not.

Making a plan can involve several different strategies. Your first idea may not give you a solution directly, but might suggest a better approach. You could look for patterns, draw more diagrams, or set up equations. You could also try to solve a similar, simpler version of the problem and see if that gives you some more insight.

If you have worked hard on the first two stages, **carrying out the plan** should be the most straightforward stage. Work out your solution according to the plan and test it. You may have to adjust the plan, but in doing so it will have given you further insights that may lead to a correct solution next time.

It is important to **look back at your solution** and reflect on what you have learned and why this plan worked when others did not. This will help you to become more confident in solving the next problem — and the one after that.

In this chapter and the next, you will learn how to solve problems using mathematical models that involve particular types of function, namely linear, quadratic, polynomial and exponential functions.

In this chapter you will learn:

- about linear functions and their graphs
- how to use linear models
- about quadratic functions and their graphs (parabolae)
- about the properties of a parabola: its symmetry, vertex and intercepts on the x-axis and y-axis
- how to use quadratic models.

RR ***It will be easier to study this chapter if you have already completed Chapters 2, 14 and 17.***

18.1 Linear models

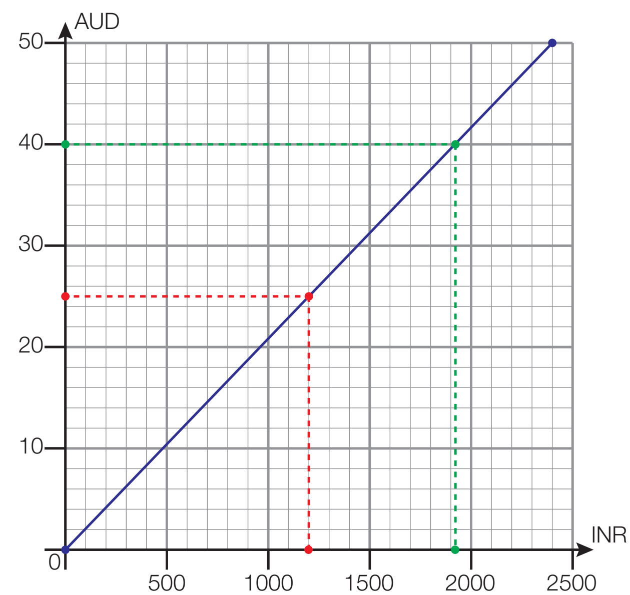
In Chapter 2, the general equation of a straight line (known as a linear equation) was given as $y = mx + c$.

After meeting function notation in Chapter 17, you now know that the same relationship can be written as $f(x) = mx + c$, to express that the input x and the output y, or $f(x)$, are related through a **linear function** whose graph is a straight line with gradient m and y-intercept c.

Many practical situations may be modelled with linear functions. For example, monthly telephone costs typically include a fixed charge plus a charge per minute; a plumber's fee is usually made up of a fixed 'call-out' charge plus further costs depending on how long the job takes.

In Chapter 4 you studied currency conversions, where each currency is linked to another via a linear function. When you are travelling abroad and using a different currency on the metro or in shops and restaurants, a straight-line graph can help you to quickly get an idea of prices.

Imagine that Logan travels from Australia to India on holiday. He changes Australian dollars (AUD) into Indian rupees (INR), and the exchange rate at the time gives him 48 INR for 1 AUD. Using this exchange rate, he draws a conversion graph that plots the number of rupees against the number of Australian dollars:



Now that Logan has this picture in his head, he can convert between INR and AUD with confidence. The graph quickly tells him, for instance, that:

- 1200 INR = 25 AUD
- 40 AUD ≈ 1900 INR.

The gradient of the graph is $m = \frac{1}{48}$ and, as 0 INR = 0 AUD, the y-axis intercept is at the origin. So the function can be written as $A(x) = \frac{1}{48}x$, where x represents the number of Indian rupees and $A(x)$ the number of Australian dollars.

Logan takes a taxi. The taxi company charges a fixed fee of 16 INR for all journeys, plus 10 INR for each kilometre travelled. The total cost of a journey is also a linear function.

If Logan travels x kilometres, he pays $10 \times x$ INR for this distance on top of the fixed charge of 16 INR, so the total cost function is $C(x) = 16 + 10x$.

Suppose Logan travels 8 km. How much will he pay?

$$C(8) = 16 + 8 \times 10 = 96 \text{ INR}$$

If Logan pays 120 INR for a taxi ride, how far has he travelled?

$$120 = 16 + 10x$$

$$120 - 16 = 10x$$

$$x = 10.4 \text{ km}$$

He visits a friend, and the friend recommends another taxi company, which has a lower initial charge but a higher charge per kilometre. This company's cost function is $D(x) = 4 + 14x$.

Logan needs to make a 20 km trip. He wants to know which of the two companies will be cheaper; he also wants to find out for what distance both companies will charge the same amount.

A graph can answer both questions. Drawing the two linear functions on a GDC makes it simple to find the **break-even point** where the charges of both companies will be the same — it is the point of intersection of the two lines:

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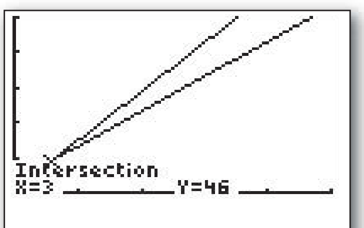

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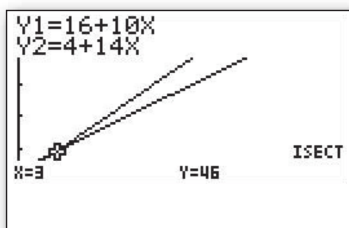

For $x = 20$:

$$C(x) = 16 + 10 \times 20 = 216 \text{ INR}$$

$$D(x) = 4 + 14 \times 20 = 284 \text{ INR}$$

so the first company will be cheaper.

Looking at the graph, the break-even point occurs at 3 km, for which the cost is 46 rupees. For journeys longer than 3 km Logan should use the first company, but for short journeys the second company would be cheaper.

Worked example 18.1

Q. Nimmi and her Theatre Studies group are presenting a show at the Iowa Festival Fringe. Nimmi is in charge of publicity and needs to order leaflets to advertise their show. A local printer quotes her '\$40 to set up and 2 cents for every leaflet'.

(a) Write down the cost function in the form $C(p) = mp + c$, where p is the number of leaflets and $C(p)$ is total cost of printing them.

(b) What is the cost of 600 leaflets?

She finds a printer on the Internet who will charge her \$32 as an initial cost, then 3 cents for every leaflet.

(c) Write down the cost function for the second printer in the form $D(p) = mp + c$.

(d) If the bill from the online printer is \$77, how many leaflets did Nimmi order?

(e) What is the break-even point where the two printers charge the same?

First, make sure you are working with consistent units. There are 100 cents in a dollar, so 2 cents = \$0.02.

A. (a) $C(p) = 40 + 0.02p$

To find the cost of 600 leaflets, substitute 600 for p in the function's formula.

(b) $C(600) = 40 + 600 \times 0.02 = 52$

so the cost is \$52.

This is similar to part (a), but with different values of m and c.

(c) $D(p) = 32 + 0.03p$

The bill from the online printer is $D(p)$, so we need to solve $D(p) = 77$.

(d) $77 = 32 + 0.03p$

$45 = 0.03p$

$p = 1500$

continued . . .

The break-even point is where $C(p) = D(p)$. Plot both functions on your GDC and look for their intersection (see '*2.1 (a) Solving linear equations using a graph*' on page 652 of the GDC chapter if you need a reminder of how to use your GDC here).

(e)

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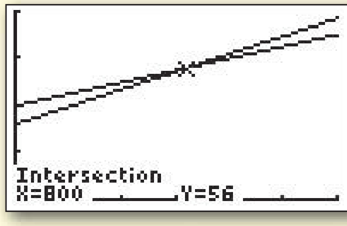

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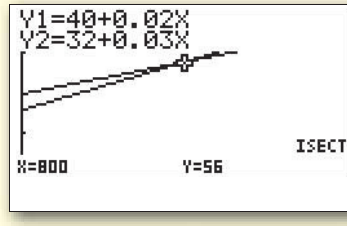

The break-even point is at 800 leaflets.

Worked example 18.2

Q. A company that makes hinges has fixed costs of €10,000. Each hinge costs €5 to make and is sold for €12.50.

(a) Write down the cost function.

(b) Write down the revenue function.

(c) Find the point at which the company starts to make a profit.

(d) What is the profit when the company makes 2400 hinges?

The cost function is made up of the fixed cost plus the cost of making the hinges at €5 per hinge.

A. (a) Let H be the number of hinges made; then they will cost €$5H$ in total to make.

$C(H) = 10000 + 5H$

The revenue is the amount earned from selling the hinges.

(b) $R(H) = 12.5H$

The company will start to make a profit when the revenue first becomes greater than the cost, so we need to look for the point where $C(H) = R(H)$, i.e. the break-even point. You can solve this equation with algebra or by plotting graphs on your GDC.

(c)

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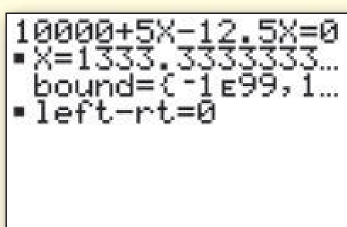

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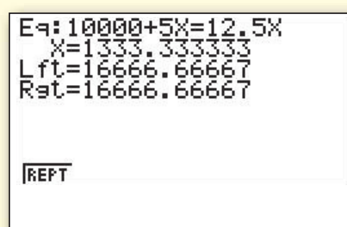


continued . . .

$10000 + 5H = 12.5H$

$10000 = 7.5H$

$H = 1333.33...$

The company starts to make a profit when it manufactures more than 1333 hinges.

The profit is the revenue minus the cost.

(d) $P(H) = R(H) - C(H) = 12.5H - (10000 + 5H)$

$P(2400) = 30000 - 22000 = 8000$

so the profit is €8000.

To find the profit from 2400 hinges, substitute 2400 for p in the profit function.

Exercise 18.1

1. The Thompsons are planning a holiday. They want to rent a family car and see the following two advertisements in a newspaper:

ZOOM CAR RENTALS

Flat fee: \$30

plus

\$0.20 per mile

SAFE RIDE RENTALS

Flat fee: \$50

plus

\$0.15 per mile

The cost of renting a car from Zoom can be expressed as:

$$C = 30 + 0.2m$$

where C is the total cost in dollars and m is the number of miles driven.

(a) Write a similar equation for the cost of renting a car from Safe Ride.

(b) What is the total number of miles that would make rental costs the same for both companies?

(c) Which of the two companies should the Thompsons use if they plan to drive at least 1000 miles? Justify your answer.

2. The Martins have switched their natural gas supply to a company called De GAS. The monthly bill consists of a standing charge of £6.50 and an additional charge of £0.035 per unit of gas used.

(a) If the Martins use n units of gas in a month, write an equation to represent the total monthly charge, $C(n)$, in pounds.

(b) Calculate their bill when the Martins use 1300 units in a calendar month.

(c) The Martins were charged £70.90 in one month. Calculate the number of units of gas they used that month.

3. The Browns are reviewing their electricity consumption. Their monthly bill includes a standing charge of £5.18 and an additional charge of £0.13 per kWh of electricity used.

(a) Assuming the Browns use n units (kWh) of energy per month, write an equation to represent the total monthly cost of electricity, $C(n)$, in pounds.

(b) In one month the Browns used 500 units of electricity. Calculate the total cost of the energy used.

(c) The electricity bill for the Browns was £89.68 in one calendar month. Determine the number of units used in that month.

4. Emma is considering the following two advertisements in the local newspaper:

GYM BUDDIES

\$120 annual subscription

plus

\$5 per visit

FIT MATES

\$300 annual subscription

plus

\$2 per visit

(a) For how many visits in a year will the cost of the two gym services be the same?

(b) Which of the two gyms should Emma use if she plans to make:

(i) no more than 52 visits a year?

(ii) at least 70 visits a year?

(c) Work out the difference in cost for (i) and (ii) in part (b).

5. Larisa and her friends are setting up a Young Enterprise company in their school to design and sell birthday cards. Initial set-up costs amount to £120. The cost of producing each card is 90 pence.

(a) Work out how many cards they have to sell to break even if each card is sold for:

(i) £1.70 (ii) £1.90 (iii) £2.10

(b) How much profit can they expect to make if they sell 300 cards at £1.90 each?

(c) How much profit would be made if they sold the first 100 cards at £1.70 each and the next 200 cards at £2.10 each?

(d) What would the estimated profit be if they sold the first 200 cards at £1.70 each and the next 100 cards at £2.10 each?

18.2 Quadratic functions and their graphs

In Chapter 2 you learned that a **quadratic equation**:

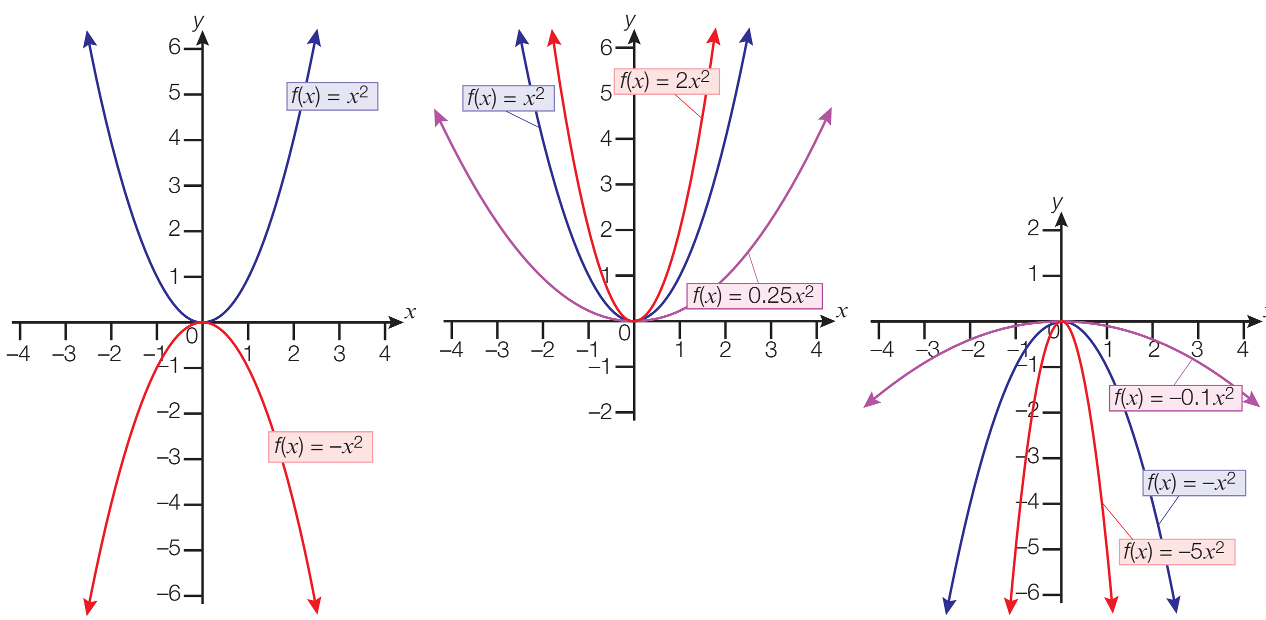
- is an equation of the general form $ax^2 + bx + c = 0$ where $a \neq 0$
- can have no solution, one solution or two possible solutions.

The concept of a **quadratic function** is broader. A quadratic function is a function having the general form:

$$f(x) = ax^2 + bx + c \text{ where } a \neq 0$$

Rather than concentrating on solutions to the equation $ax^2 + bx + c = 0$, as we did in Chapter 2, we will now look at other properties of the function $f(x) = ax^2 + bx + c$, and discuss how to use such a function in mathematical modelling.

Using your GDC, draw graphs of $f(x) = ax^2$ for different values of a. Try both positive and negative values. (See '*22.2G Graphs*' on page 645 of the GDC chapter for a reminder of how to plot graphs if you need to.)

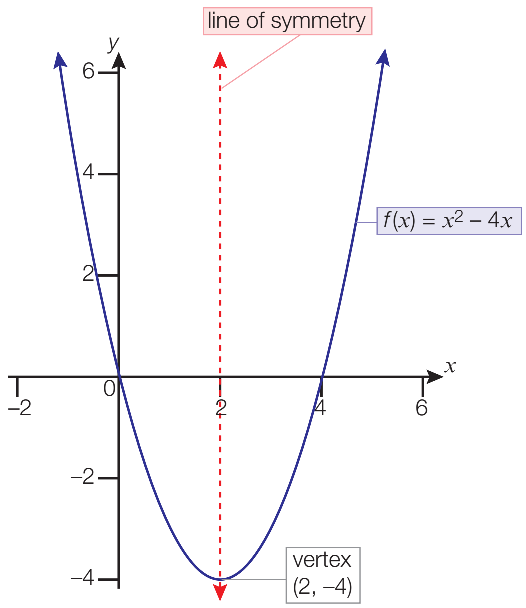


Compare your graphs. What do you notice?

You should see that:

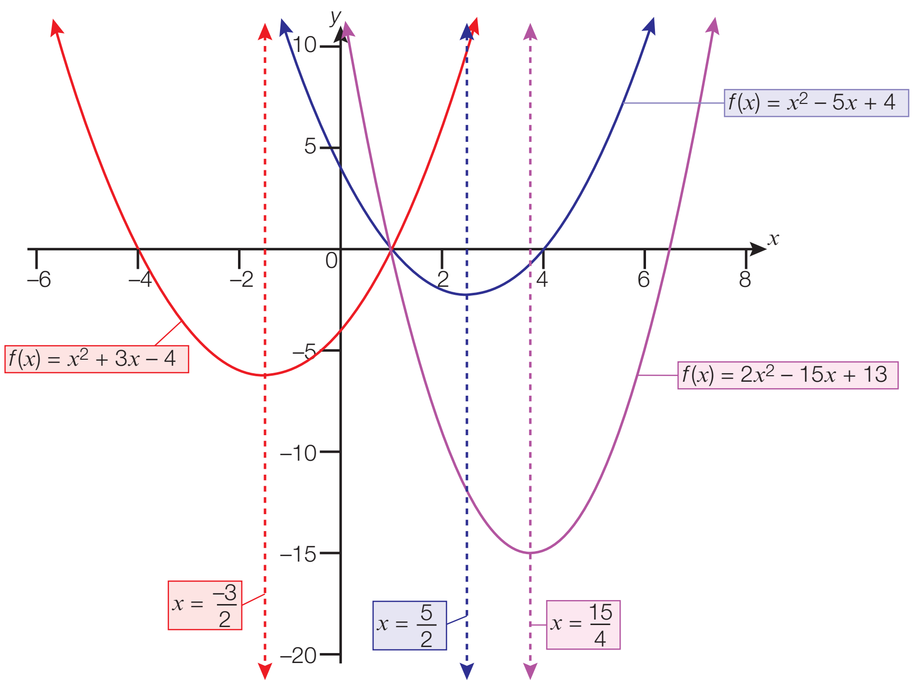
- If a is positive, the function has a minimum point.
- If a is negative, the function has a maximum point.
- As the magnitude of a increases (whether it is positive or negative), the parabola becomes steeper.
- As the magnitude of a decreases, the parabola becomes shallower.

You should also have observed that the graph is symmetrical: there is a **line of symmetry** (also called an axis of symmetry) that cuts the curve in half so that each side is the mirror image of the other. The line of symmetry runs through the maximum or minimum point. This turning point, where the curve changes direction, is called the **vertex** of the parabola.



Equation of the line of symmetry

Using your GDC, draw graphs of $f(x) = ax^2 + bx + c$ with different values of a and b. See if you can find a link between the values of a and b that will give you an equation for the line of symmetry.



Function	$f(x) = x^2 - 5x + 4$	$f(x) = x^2 + 3x - 4$	$f(x) = 2x^2 - 15x + 13$
Location of line of symmetry	$x = \frac{5}{2}$	$x = \frac{-3}{2}$	$x = \frac{15}{4}$

hint

Remember that the coefficient is the number that multiplies a variable.

In the examples above, look at the **coefficient** of the x^2 term and the coefficient of the x term.

In each case, if you take the coefficient of the x term (e.g. -5), change its sign (e.g. 5), divide by the coefficient of the x^2 term (e.g. 1) and then divide again by 2, you get the location of the line of symmetry (e.g. $x = \frac{5}{2}$).

This gives us a general formula for the line of symmetry of a quadratic function.

$a=\pi r^2$ For the quadratic function $f(x) = ax^2 + bx + c$, the equation of the line of symmetry is $x = \frac{-b}{2a}$.

The vertex of a parabola

As the vertex of the parabola lies on the line of symmetry, you can use the line of symmetry as the x-coordinate to calculate the corresponding y-coordinate of the vertex.

For example, $f(x) = x^2 - 5x + 4$ has line of symmetry $x = \frac{5}{2}$.

Substituting this value of x into the function, we get:

$$y = f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 4 = -\frac{9}{4}$$

So the vertex is the point $\left(\frac{5}{2}, -\frac{9}{4}\right)$.

You could also find the vertex and line of symmetry of a parabola using your GDC, but in that case you would work the other way round, locating first the vertex and then the line of symmetry. See '*18.1 Using a graph to find the vertex and line of symmetry of a parabola*' on page 680 of the GDC chapter if you need a reminder of how.

For example, to find the vertex of $f(x) = 4 + 6x - x^2$, draw the graph of $y = 4 + 6x - x^2$ on your GDC, and find the coordinates of the maximum point.

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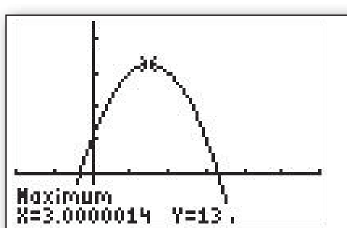

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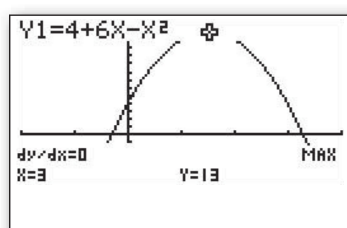

The maximum point is at (3, 13), so the equation of the line of symmetry is $x = 3$.

Worked example 18.3

Q. Draw the function $f(x) = 2x^2 - 7x + 8$ on your GDC.

(a) Write down the coordinates of the vertex.

(b) Write down the equation of the line of symmetry.

(c) Write down the range of the function.

RR ***If you need a reminder of what the range is, look back at section 17.1.***

(d) What is the range of the function $-f(x)$?

A.

Enter $y = 2x^2 - 7x + 8$ into your GDC and draw its graph.

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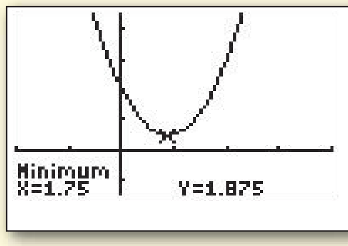

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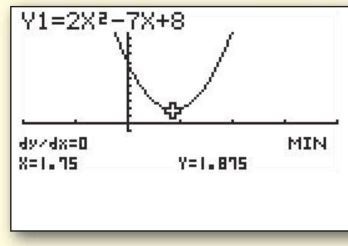

Use your GDC to find the coordinates of the minimum point. See section '18.1' on page 680 of the GDC chapter if you need to.

(a) The vertex is at (1.75, 1.875).

The equation of the line of symmetry is given by the x-coordinate of the vertex.

(b) The line of symmetry is $x = 1.75$.

Check using the formula $x = \frac{-b}{2a}$.

$$x = \frac{-(-7)}{2 \times 2} = 1.75$$

As the minimum is at (1.75, 1.875), the graph never has a y-value less than 1.875, though it can extend arbitrarily far upwards.

(c) The range is $f(x) \geq 1.875$.

continued . . .

(d)

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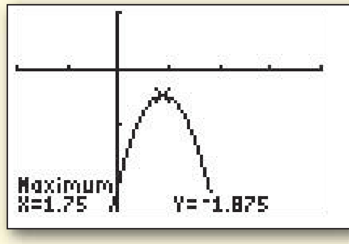
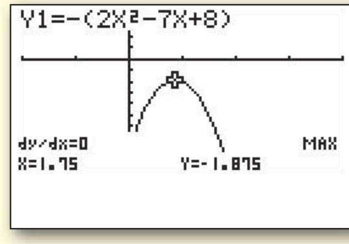


CASIO



$-f(x) \leq -1.875$

Draw the graph of $-f(x)$ on your GDC by entering $y = -(2x^2 - 7x + 8)$. You can see that this is just the graph of $f(x)$ reflected in the x-axis, so the minimum has now become a maximum.

Exercise 18.2A

1. Draw the following functions on your GDC. In each case:

(i) Write down the coordinates of the vertex.

(ii) State the equation of the line of symmetry.

(iii) Write down the range of the function.

(a) $f(x) = x^2 + 3x$

(b) $f(x) = x^2 - 7x + 2$

(c) $f(x) = -2x^2 + x - 6$

(d) $g(x) = 9 - 4x + 3x^2$

(e) $g(x) = (x - 3)(x + 5)$

(f) $g(x) = 1 - (x + 1)(2 - x)$

(g) $h(x) = \frac{1}{2}x^2 - \frac{3}{4}x + \frac{5}{2}$

(h) $f(x) = 5.3 + 4.9x - 0.7x^2$

2. By sketching the graph of each of the following quadratic functions, find the coordinates of the vertex and hence:

(i) state the maximum or minimum value of the function

(ii) determine the equation of the line of symmetry

(iii) state the range of the function.

(a) $f(x) = 1.44 - 0.1x^2$

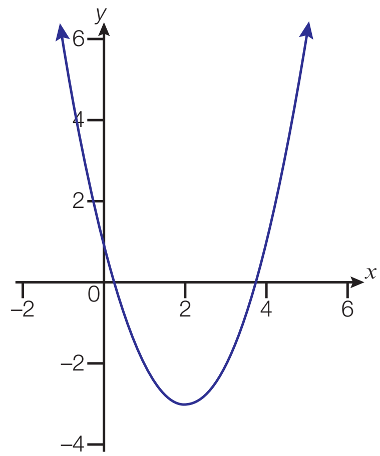
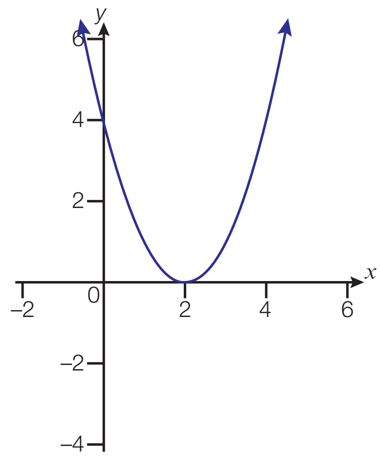
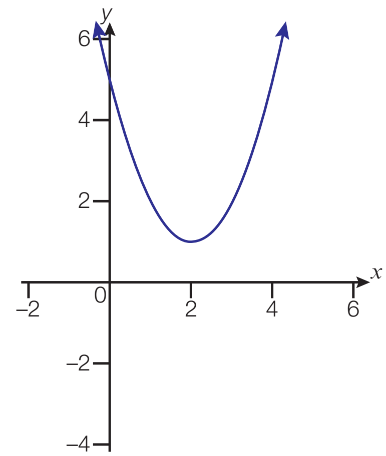
(b) $g(x) = 10 - (5 - 2x)^2$

(c) $h(x) = 3(x + 1.7)^2 + 8$

Intercepts on the x and y axes

The quadratic function $f(x) = ax^2 + bx + c$ will always have **one *y*-intercept** — it is the value of $f(x)$ when $x = 0$, which is just the number c.

With ***x*-intercepts**, there are three different situations that can arise.

(a)	(b)	(c)
		
The curve cuts the x-axis at two points.	The curve touches the x-axis at one point.	The curve does not touch or cross the x-axis.

In (a), there are two places where the function crosses the x-axis; so there are two x-intercepts. If you are using the graph to solve a quadratic equation, you can say that there are two solutions to the equation, or two real **roots**.

In (b), the parabola just touches the x-axis tangentially at one point, so there is only one solution to the equation $ax^2 + bx + c = 0$.

FF **Tangentially means that the curve only touches the axis at a single point and does not cross the axis. You will learn about tangents in Chapter 20.**

In (c), the curve does not meet the x-axis at all, so there are no x-intercepts and thus no solutions to the equation $ax^2 + bx + c = 0$.

After drawing the graph of a quadratic function $f(x) = ax^2 + bx + c$ on your GDC, you can easily use it to locate the solutions of $ax^2 + bx + c = 0$ (i.e. the x-intercepts); see '*18.2 Using a graph to find the zeros (roots) of a quadratic equation*' on page 681 of the GDC chapter if you need a reminder of how to do this.

GDC

Some calculators use the word 'zero' instead of 'solution' or 'root'.

Worked example 18.4

Q. Use your GDC to draw the following functions:

$f(x) = 3x^2 - 2x + 1, \qquad g(x) = 5 + x - 2x^2, \qquad h(x) = x^2 + 4x$

(a) Write down the y-intercept of $h(x)$.

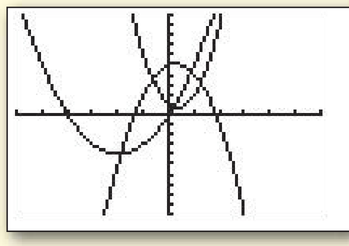
(b) Which function(s) has/have two x-intercepts?

(c) Which function has no x-intercept?

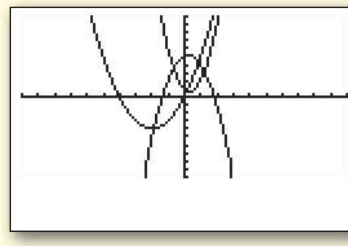
(d) Consider the function $j(x) = x^2 + 4x + 4$. How many x-intercepts does this function have? Write down the value(s) of x at the intercept(s).

A.

TEXAS



CASIO



If all three graphs are plotted on the same axes, you can tell them apart by observing that $g(x)$ is the curve with a maximum point, and $h(x)$ is the curve that goes through (0, 0).

(a) The y-intercept of $h(x)$ is (0, 0).

The y-intercept is the point on the graph where $x = 0$.

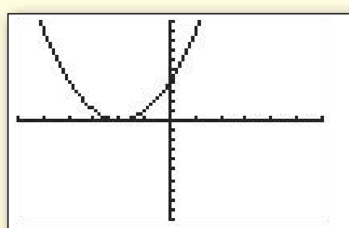
(b) $g(x)$ and $h(x)$ have two x-intercepts.

(c) $f(x)$ has no x-intercept.

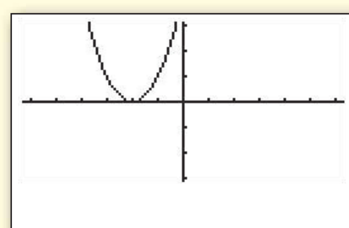
The minimum point of $f(x)$ is above the x-axis, so the curve will never touch or cross the x-axis.

(d)

TEXAS



CASIO



$j(x)$ has one x-intercept at $x = -2$.

Draw the graph of $j(x)$. It touches the x-axis at the point (–2, 0). Alternatively, note that $j(x) = h(x) + 4$, so the graph of $j(x)$ is just the graph of $h(x)$ shifted up by 4 units; the vertex (–2, –4) of $h(x)$ becomes the vertex (–2, 0) of $j(x)$ on the x-axis.

Exercise 18.2B

1. Find the coordinates of the points where the graph $y = x^2 - 2x - 7$ intersects the x-axis.

2. Consider the following functions:

   $f(x) = 6x^2 - 13x - 5, \quad g(x) = x^2 + 3x + 5, \quad h(x) = \frac{16}{9}x^2 - \frac{20x}{3} + \frac{25}{4}$

   (a) Which of the three functions from above has:

   (i) two x intercepts? (ii) no x intercepts? (iii) one x intercept?

   (b) State the y-intercept of each of the three functions.

3. Draw the graphs of the following quadratic functions on your GDC, and in each case state the coordinates of:

   (i) the x-intercepts (ii) the y-intercept

   (a) $y = 3x^2 - 12$ (b) $y = 16 - 9x^2$

   (c) $y = 8 - x(x - 5)$ (d) $y = (x - 3)^2 - (1 - 2x)^2$

   (e) $y = 1 + (2x - 7)(x + 3)$ (f) $f(x) = 3.144 - 1.07(2x - 5)^2$

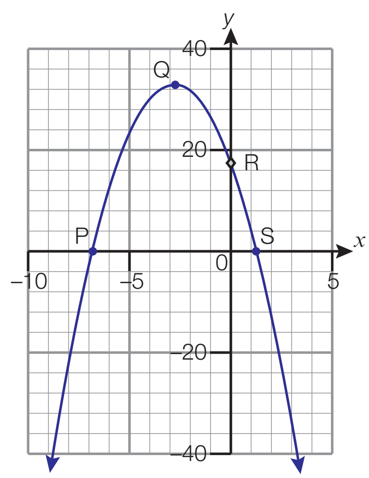
4. Consider the function $f(x) = 7x - 1.4x^2, -1 \le x \le 5.5$.

   (a) State the x-intercepts.

   (b) Write down the equation of the line of symmetry.

   (c) State the range of $f(x)$.

5. The diagram shows the graph of the function $f(x) = 18 - 11x - 2x^2$. Q is the vertex of the parabola; P, R and S are the intercepts with the axes.



   (a) Use your GDC to find the coordinates of points P, Q, R and S.

   (b) State the range of the parabola.

   (c) Write down the equation of the line of symmetry.

GDC

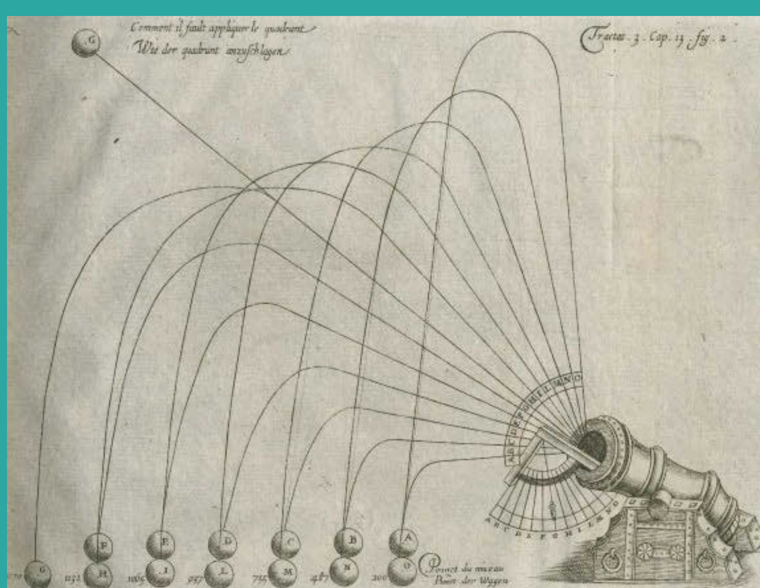
If you are taking values from a sketch on your GDC, look carefully at the scale used for the window. If you set a scale of 1, you can count along the axes to get the x and y values. However, if you have used ZOOM FIT (TEXAS) or AUTO (CASIO), the GDC will set the scale, which is often diverse and complicated, with fractional scales, giving you a very different picture.

18.3 Quadratic models

Look around you. How many parabolas have you encountered today? Have you walked past a fountain or travelled over a suspension bridge? A quadratic function can be used to model all of these shapes.

Ask two friends to throw a ball to each other, and watch the path of the ball or, if you can, take a video of the ball-throwing and play it back frame by frame. The path of the ball looks like a parabola.

Galileo Galilei (1564–1642) was born in Pisa, Italy. He began to study medicine at the University of Pisa when he was 17, but did not complete his studies there, leaving in order to concentrate on philosophy and mathematics. In 1589, the university appointed him professor of mathematics, and a year later he wrote his book *De Moto* on the study of motion. At the time, with the development of guns and cannons, there was much interest in understanding the motion of projectiles. Galileo used a combination of experiments and mathematics to investigate how horizontal and vertical forces determine the path of a projectile; he showed that the path of a projectile can be modelled by a quadratic equation.



Quadratic functions can be used to solve problems in many real-life situations. Using a quadratic function to solve a problem is quite straightforward if you are given the function.

You learned about solving quadratic equations in Chapter 2.

Worked example 18.5

Q. The height of a ball that has been thrown up into the air can be modelled by the quadratic function $H(t) = a + bt - 4.9t^2$ where t is the time in seconds after the ball is thrown, a is the initial height of the ball in metres and b is the initial velocity of the ball.

(a) Amy is playing with a tennis ball. When the ball is at a height of 1.2 m above the ground she hits it vertically upwards with an initial velocity of 22 m s^{-1}. What is the ball's height after 2 seconds?

(b) Catherine can hit the ball harder. She hits the ball vertically upwards with an initial velocity of 26 m s^{-1} from a height of 1.5 m. When the ball reaches a height of 35.9 m, how long has it been in the air?

A. (a) For Amy, $a = 1.2$ and $b = 22$

$H(t) = 1.2 + 22t - 4.9t^2$

Substitute given values of a and b into the expression for $H(t)$.

$H(2) = 1.2 + 22 \times 2 - 4.9 \times 2^2 = 25.6$ m

To find the height after 2 seconds, substitute $t = 2$ in the function $H(t)$.

(b) For Catherine, $a = 1.5$ and $b = 26$

$H(t) = 1.5 + 26t - 4.9t^2$

Find the function for Catherine in the same way.

$1.5 + 26t - 4.9t^2 = 35.9$

If the height is 35.9 m, then $H(t) = 35.9$, and we need to solve this equation for t.

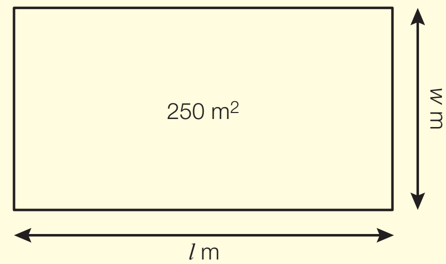
From GDC: $t = 2.79$ seconds

Use any method you like, for example the equation solver on your GDC; see '2.3 Solving quadratic equations' on page 656 of the GDC chapter if you need a reminder.

Using a quadratic function to solve a problem is more complicated if you have to find an expression for the function first.

Worked example 18.6

Q. The area of a rectangular vegetable garden is 250 m². The perimeter of the plot is 65 m. What are the dimensions of the garden?



A.

First, express the given information using algebra. The perimeter is made up of two lengths plus two widths.

$P = 2w + 2l = 65$

$w + l = 32.5$

so $l = 32.5 - w$

The area of the garden is length times width. We now have a quadratic equation.

$A = l \times w = 250$

$(32.5 - w)w = 250$

$32.5w - w^2 = 250$

Now we can find w by solving this quadratic equation. First rearrange it into the general form.

$w^2 - 32.5w + 250 = 0$

Use your GDC to solve (see Section '*2.3 Solving quadratic equations*' on page 656 if you need to.)

From GDC: $w = 12.5$ or $w = 20$

Calculate the corresponding values of l.

If $w = 12.5$, then $l = 32.5 - 12.5 = 20$

If $w = 20$, then $l = 32.5 - 20 = 12.5$

Notice that either of the two solutions for w will give the same overall set of dimensions.

The dimensions are 12.5 m × 20 m.

There is another way that you can investigate quadratic functions to model parabolas that you see in everyday life. Your GDC can fit a curve to any coordinates that you give it, and find the equation that fits your data the closest. It can also give you the product moment correlation coefficient, so that you know how accurate the equation is as a model for your data. See '*18.3 Using the statistics menu to find an equation*' on page 682 of the GDC chapter for a reminder if you need to.

RR **You learned about product moment correlation coefficients in Chapter 12.**

Worked example 18.7

Q. Kitty sees an arch that she thinks is in the shape of a parabola. She takes a photograph, and then lays a grid over the picture to read off some coordinates. She finds the following coordinates:

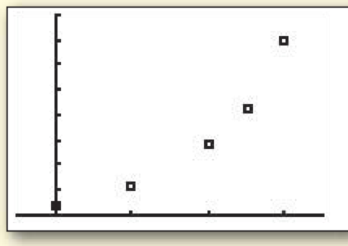
x	0	2	4	5	6
y	0.4	1.2	2.8	4.2	7.0

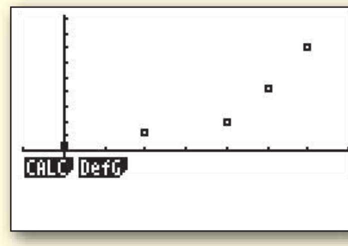
Use this table of coordinates to determine if Kitty is correct; is the arch in the shape of a parabola?

Enter the coordinates into your GDC and draw a scatter graph. (See '*12.1 Drawing a scatter diagram of bivariate data*' on page 672 of the GDC chapter if you need a reminder of how to draw a scatter diagram on your GDC.)

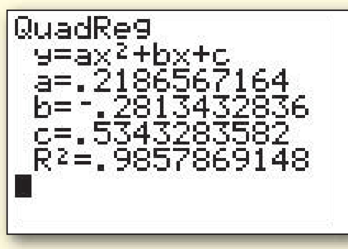
A.

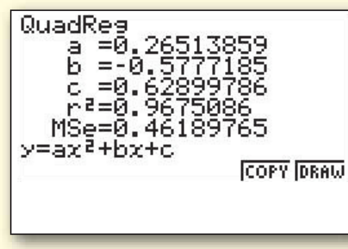
TEXAS | **CASIO**

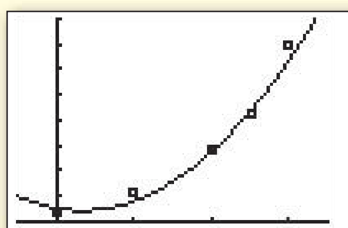


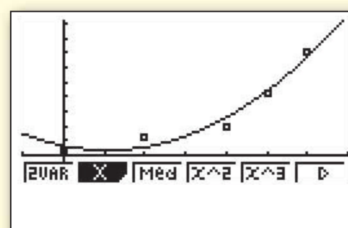


The scatter graph confirms Kitty's idea because it shows a positive correlation between the x and y coordinates; use your GDC to fit a curve to the points and find an equation of the curve. (See '*18.3 Using the statistics menu to find an equation*' on page 682 of the GDC if you need to.)









Equation of the curve:

$y = 0.22x^2 - 0.28x + 0.53$

$r = 0.99$ (2 d.p.)

Equation of the curve:

$y = 0.27x^2 - 0.58x + 0.63$

$r = 0.98$ (2 d.p.)

The value of r suggests there is a strong positive correlation between the x- and y-coordinates and the equation of the curve calculated by the GDC. The equation of the curve contains x^2 and is therefore a quadratic, which suggests that Kitty was correct in thinking the arch is in the shape of a parabola.

hint

You will see that the two GDCs have given different results for the coefficients of the quadratic function, even though both curves seem to fit quite well; this is because they are using different formulae to calculate the quadratic curve of best fit. If you use this curve-fitting technique in a project, it might be an issue to discuss when you are assessing the validity of your results.

It has been said that 'quadratic equations underpin all modern science'. Is this true for other fields of study too? Think about economics, or architecture and the principles of design.

Using the same method, you could fit a quadratic function to the path of the ball thrown between two people that we mentioned at the beginning of this section.

Exercise 18.3

1. Sabina hits a tennis ball vertically upwards from an initial height of 1.4 metres giving the ball an initial velocity of $20\,\text{m}\,\text{s}^{-1}$.

The flight of the ball can be modelled with the equation:

$$h(t) = 1.4 + 20t - 4.9t^2$$

where $h(t)$ is the height of the ball above the ground, t seconds after Sabina hit it.

(a) Find the height of the ball after 1.5 seconds.

(b) Use your GDC to draw the graph of $h(t)$ and hence find the maximum height of the ball above the ground during its flight.

(c) Sabina catches the ball when it falls back to a height of 1.4 metres. Find the total time that the ball was in flight.

2. Joey reckons that the approximate distance it takes to stop a car, depending on the speed at which the car is travelling, can be modelled by the equation:

$$y = 0.0555x^2 + 1.112x - 0.6494$$

where y is the stopping distance in feet and x is the speed of the car in miles per hour.

Use the model to predict estimates of the stopping distances in the following table to 1 d.p.

Speed, x (mph)	20	30	40	50	60	70
Stopping distance, y (feet)						

3. The profit made by a large Slovenian company can be modelled as a function of the expenditure on advertising:

$$P(x) = 40.8 + 3.6x - 0.2x^2$$

where P is the profit in millions of euros and x is the expenditure on advertising in millions of euros.

(a) Use your GDC to sketch the graph of $P(x)$ for $0 \leq x \leq 30$.

(b) Hence find the expenditure on advertising that will maximise the profit.

(c) Calculate the estimated profit when the company spends €5 million on advertising.

4. Tom hits a golf ball. The height of the ball can be modelled by the equation:

$$h(t) = 7t - t^2$$

where $h(t)$ is the height of the ball in metres, t seconds after Tom hit it.

(a) What was the height of the ball after 3 seconds?

(b) After how long did the ball reach a height of 10 metres for the first time?

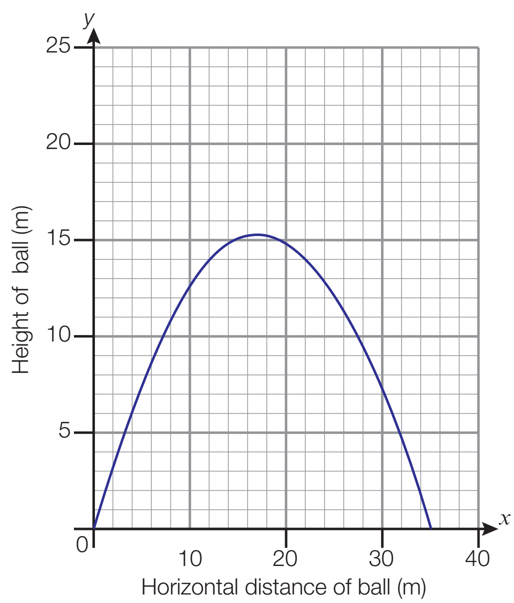
(c) What was the maximum height of the ball?

(d) For how long was the ball more than 8 metres above the ground?

5. In a game of cricket, the batsman hits the ball from ground level. The flight of the ball can be modelled by a parabola. The path of the ball is expressed as:

$$y = 1.732x - 0.049x^2$$

where y is the height of the ball in metres above the ground and x is the horizontal distance in metres travelled by the ball.



(a) Use your GDC to find:

(i) the maximum height reached by the ball

(ii) the horizontal distance travelled by the ball before it falls to the ground.

(b) Work out the height of the ball when:

(i) $x = 12$ m (ii) $x = 20$ m.

(c) Find the horizontal distance of the ball from the batsman when the ball first reaches a height of 13 metres.

Summary

You should know:

- how to identify linear functions, $f(x) = mx + c$, and their graphs
- how to use linear models for solving practical problems
- how to identify quadratic functions, $f(x) = ax^2 + bx + c$, and their graphs (parabolae)
- about the various properties of a parabola:
  - the equation of its line of symmetry, $x = \frac{-b}{2a}$
  - how to find the coordinates of its vertex
  - how to find its intercepts on the x-axis and y-axis
- how to use quadratic models to solve practical problems.

Mixed examination practice

Exam-style questions

1. Draw the graph of the following functions on your GDC. In each case:

(i) Write down the coordinates of the vertex.

(ii) State the equation of the line of symmetry.

(iii) Write down the range of the function.

(a) $y = x(5x - 9)$ (b) $y = 7 - 2x^2$ (c) $y = x^2 - 3x - 28$

2. Sketch the graph of $f(x) = 11 - 4x - 6x^2$ on your GDC.

(a) Find the maximum or minimum value of the function.

(b) Determine the equation of the line of symmetry.

(c) State the range of the function.

3. The Smiths are looking for a plumber. They find this advert in the local paper.

Sydner Plumber
\$140 call out + \$98 per hour

The total charge, C, for plumbing services can be written as:

$C = a + bx$

where x is the number of hours taken to finish the work.

(a) State the values of a and b.

(b) How much will it cost the Smiths if it takes 5 hours to complete the work?

(c) In his next call out, the plumber charged Mr Jones \$826. How long did the plumber work on this job?

4. The area of a rectangular soccer pitch is 4050 m². The perimeter of the pitch is 270 m.

(a) Taking the length of the pitch to be x m, find an expression for the width of the pitch in terms of x.

(b) Hence show that x satisfies the equation $x^2 - 135x + 4050 = 0$.

(c) Solve the equation to find the dimensions of the pitch.

5. The trajectory of a golf ball is in the form of a parabola. The path of the ball can be modelled by the equation:

$y = 0.57x - 0.002375x^2$

where x is the distance in metres from where the ball was hit and y is the height of the ball in metres above the ground.

(a) Find the maximum height reached by the ball.

(b) How far does the ball travel before it hits the ground for the first time?

(c) What was the horizontal distance of the ball from where it was hit when its height was 19 m?

6. The subscription to a mathematics revision website MATHSMANAGER is £5 per month. There is an additional charge of 60p per visit. The total monthly cost of using the site can be represented as:

$$C = C_0 + n \times t$$

where: C is the total monthly cost

C_0 is the fixed monthly subscription fee

n is the number of visits in a month

t is the charge per visit.

(a) Calculate the total cost for:

(i) Saif, who visited the site 11 times last month

(ii) Jeevan, who made 40 visits to the site last month.

A second website MATHS-PLUS-U charges a £3.20 monthly subscription and 65p per visit.

(b) How much would it have cost Saif and Jeevan individually if they had used MATHS-PLUS-U instead?

(c) Jack intends to revise intensely over the two months preceding his mock examinations. He is planning to visit one of the revision websites at least 100 times.

(i) Which of the two sites will be cheaper for him to use?

(ii) What is the difference in cost between using these sites?

Past paper questions

1. The function $Q(t) = 0.003t^2 - 0.625t + 25$ represents the amount of energy in a battery after t minutes of use.

(a) State the amount of energy held by the battery immediately before it was used.

(b) Calculate the amount of energy available after 20 minutes.

(c) Given that $Q(10) = 19.05$, find the average amount of energy produced per minute for the interval $10 \le t \le 20$.

(d) Calculate the number of minutes it takes for the energy to reach zero.

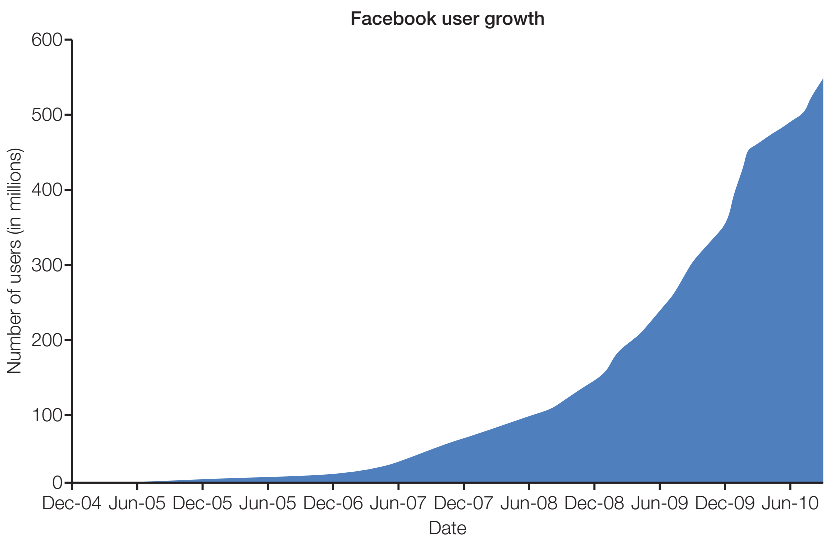
[Total 6 marks]

[May 2006, Paper 1, TZ0, Question 7] (© *IB Organization 2006*)

See also the past paper questions at the end of Chapter 2.

Chapter 19 Exponential and polynomial functions

'Exponential growth' is a phrase we hear every day on the news and in general life. But what does exponential mean? And what is exponential growth? The graph below shows the increase (in millions) in number of users of the social network site *Facebook* over a five and a half year period. It demonstrates the typical shape of an exponential function: the increase in the vertical axis starts off very small and gradual and then increases in massive jumps of size despite the changes in the horizontal axis maintaining the same size. The graph below demonstrates that *Facebook* has seen an exponential growth in its number of users from 2004 to 2010.



An exponential function gets its name from the fact that the independent variable is an exponent in the equation and this causes the dependent variable to change in very large jumps.

In this chapter you will learn:

- about exponential functions and their graphs
- how to find horizontal asymptotes for exponential functions
- how to use exponential models
- about functions of the form $f(x) = ax^n + bx^m + \ldots$, where n and m are positive integers, and their graphs
- how to use models of the form $f(x) = ax^n + bx^m + \ldots$ where $n, m \in \mathbb{Z}^+$
- how to use graphs to interpret and solve modelling problems.

RR ***It will be easier to study this chapter if you have already completed Chapters 2, 14, 17 and 18.***

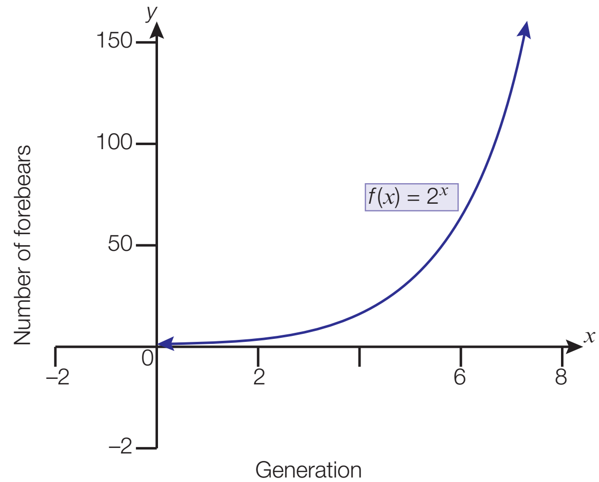
19.1 Exponential functions and their graphs

Living organisms that reproduce sexually have two biological parents, four biological grandparents and eight great-grandparents. So the number of forebears (ancestors) doubles with each generation.

You can demonstrate this example using a table:

Generation	1	2	3	4	5	6	x
Number of forebears	2	4	8	16	32	64	2^x
Power of 2	2^1	2^2	2^3	2^4	2^5	2^6	2^x

You can show the same information using a graph:



You can see from the graph that the growth in the number of forebears is very fast. Human genealogists use an estimate of four generations per century, so if you try to trace your family back 200 years, you could be looking for $2^8 = 256$ different people!

Some organisms reproduce very fast, such as bacteria; they can achieve a population of millions in a very short time.

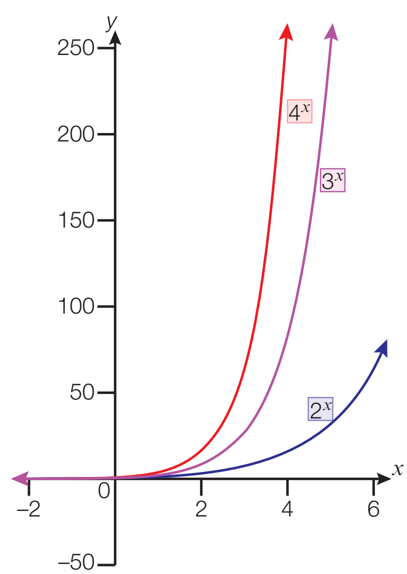
The type of growth described by the table and graph above is called **exponential growth**, and it is based on the function $f(x) = a^x$ where a is a number greater than 1.

The general form of an **exponential function** is:

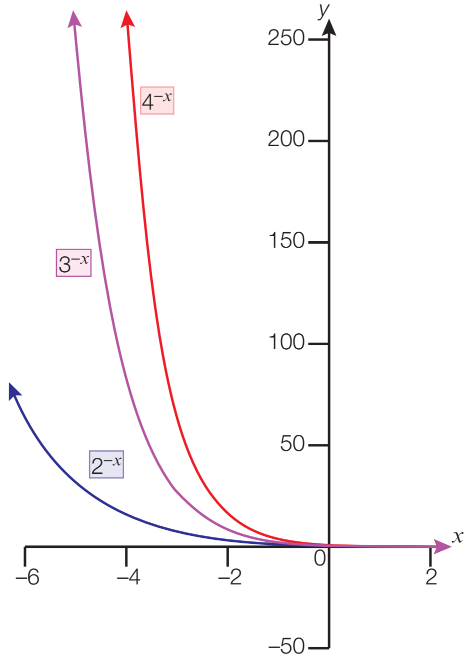
$$f(x) = ka^x + c$$

where $a > 0$, x is a variable that can be positive or negative, and k and c are constants (which can be positive or negative).

The exponential function is very interesting to explore using a computer graphing package or a GDC. Start with the simplest case and draw graphs of $f(x) = a^x$ for different values of $a > 1$:

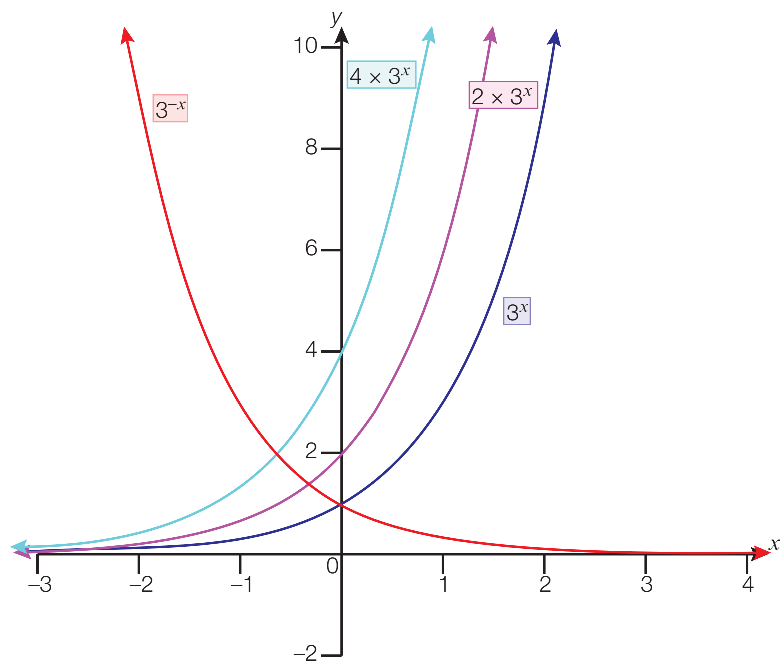


Next, draw the $f(x) = a^{-x}$ (note the negative sign in the exponent) for different values of $a > 1$:

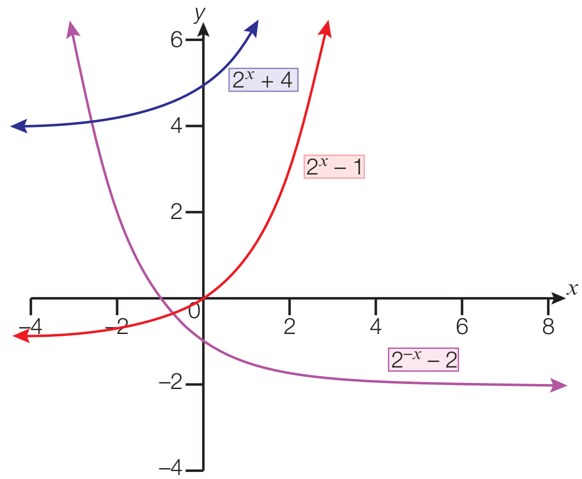


Note that this is the same as drawing graphs of $f(x) = a^x$ for positive values of $a < 1$. Try it: draw the graphs of a^x for $a = 0.5, 0.333, 0.25$. Can you see why this is the case?

You can also draw a set of graphs of $f(x) = ka^x$ and $f(x) = ka^{-x}$ for the same value of a but different values of k:



Now try drawing the graphs of $f(x) = a^x \pm c$ with the same value of a but different values of c.



After these explorations, you should find that:

- The curve always has a similar shape.
- The curve levels off to a particular value. The horizontal line at this y-value is called a **horizontal asymptote**; the curve approaches this line very closely but never actually reaches it.
- The graph of $f(x) = a^x$ or a^{-x} passes through the point (0, 1).
- The graph of $f(x) = ka^x$ or ka^{-x} passes through the point (0, k).
- The constant $\pm c$ moves the curve of ka^x up or down by c units, giving the graph a horizontal asymptote at c or $-c$.

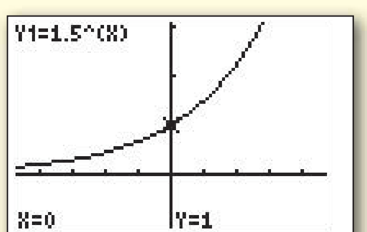
RR *You learned about asymptotes in Chapter 17.*

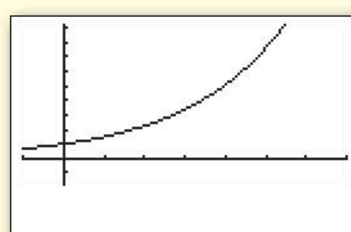
Worked example 19.1

Q. Draw the following functions on your GDC and find the equation of the asymptote for each function.

(a) $f(x) = 1.5^x$ (b) $g(x) = 3^x - 2$ (c) $h(x) = 3 + 2^{-x}$

A. (a) TEXAS CASIO





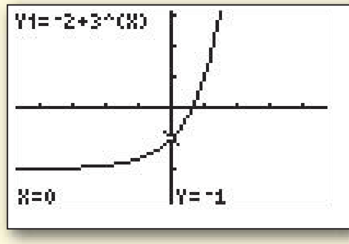
The asymptote is $y = 0$ (the x-axis).

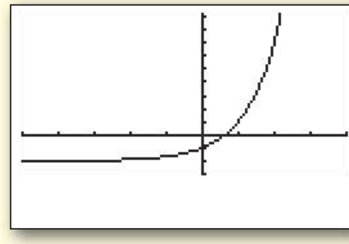
GDC

See '*17.3 Finding the horizontal asymptote*' on page 680 of the GDC chapter if you need a reminder of how to find horizontal asymptotes with your GDC.

continued . . .

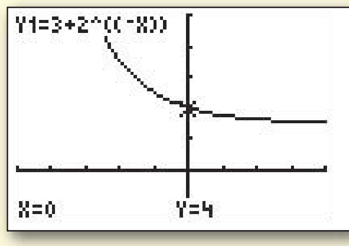
(b) TEXAS CASIO

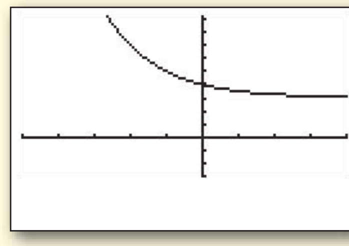




The asymptote is $y = -2$.

(c) TEXAS CASIO





The asymptote is $y = 3$.

Evaluating exponential functions

Recall from section 17.2 that function notation provides an efficient shorthand for substituting values into a function. For example, to find the value of $f(x) = 6 + 3^{-x}$ when $x = 2$, you write $f(2)$ to mean 'replace x by 2 in the expression $6 + 3^{-x}$'. This is called '**evaluating** the function $f(x)$ at $x = 2$'.

You can evaluate any function at a given value using either algebra or your GDC. If you use your GDC, be careful to enter brackets where needed, especially with negative values.

Let's look at how to evaluate $f(x) = 6 + 3^{-x}$ at $x = 2$ with three different methods.

Using algebra:

Replace x with 2.

$$f(2) = 6 + 3^{-2} = 6 + \frac{1}{3^2} = 6\frac{1}{9} = \frac{55}{9}$$

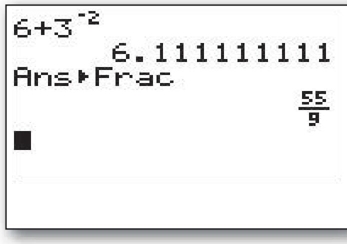
RR See Learning links 1B on page 24 for a reminder about negative indices if you need to.

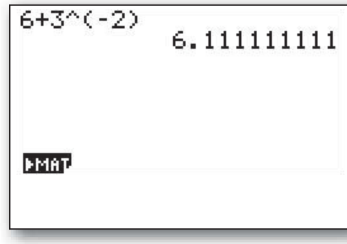
exam tip

Using algebra is sometimes quicker and more accurate than using a GDC. If you give your answer as a fraction, the answer is exact, whereas a recurring decimal is not.

Using your GDC to evaluate directly:

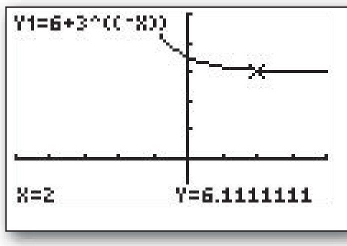
TEXAS **CASIO**

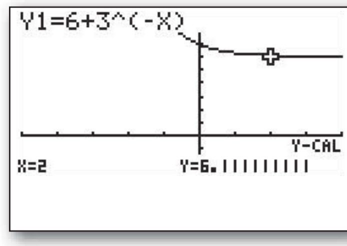




Using a graph on your GDC (see '*19.2 Solving unfamiliar equations*' on page 684 of the GDC chapter, if you need a reminder).

TEXAS **CASIO**





If you are drawing an exponential graph on a GDC, you need to be particularly careful in your choice of scale for the axes. In this example, $f(x) \geq 6$ always, so there is no need for a negative scale on the y-axis; however, you will only get a good view of the shape of the graph if you allow a wide enough domain for the x-axis.

Check that the same methods give $f(-1) = 9$ and $f(-3) = 33$.

Worked example 19.2

Q. Let $f(x) = -1 - 2^{-x}$, $g(x) = 1 + 3^x$ and $h(x) = 2 \times 1.5^{-0.5x}$. Find:

(a) $f(2)$ (b) $g(5)$ (c) $h(-4)$

Let's calculate these using algebra first.

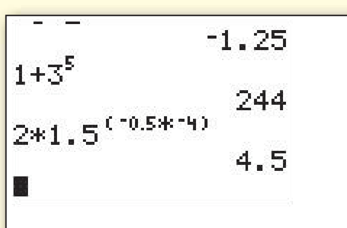
A. (a) $f(2) = -1 - 2^{-2} = -1 - \dfrac{1}{2^2} = -1 - \dfrac{1}{4} = -\dfrac{5}{4}$

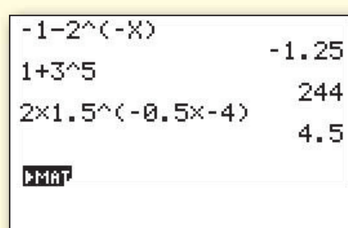
(b) $g(5) = 1 + 3^5 = 1 + 243 = 244$

(c) $h(-4) = 2 \times 1.5^{-0.5 \times (-4)} = 2 \times 1.5^2 = 2 \times 2.25 = 4.5$

Now check the answers using your GDC.

TEXAS **CASIO**





Exercise 19.1

1. Draw the following functions on your GDC, and find the equation of the asymptote in each case:

   (a) $f(x) = 3^x$ (b) $f(x) = 3^{2x}$

   (c) $f(x) = 3^x + 5$ (d) $f(x) = 3^x - 4$

   (e) $f(x) = 2 \times 3^x - 4$

2. Draw the following functions on your GDC for the given domain, and in each case state the coordinates of the y-intercept and the range.

   (a) $g(x) = 5^x$ for $-3 \leq x \leq 6$ (b) $g(x) = 5^{3x}$ for $-4 \leq x \leq 4$

   (c) $g(x) = 5^{3x+2}$ for $-2 \leq x \leq 5$ (d) $g(x) = 5^{x+7}$ for $-10 \leq x \leq 0$

   (e) $g(x) = 5^{x-4}$ for $0 \leq x \leq 9$

3. Draw the following functions on your GDC. In each case state:

   (i) the equation of the asymptote

   (ii) the coordinates of the y-intercept

   (iii) the range.

   (a) $f(x) = 2^{-x}$ for $x \geq -4$ (b) $f(x) = 3^{-x}$ for $x \geq -6$

   (c) $f(x) = \left(\frac{1}{2}\right)^x$ for $x \geq -8$ (d) $f(x) = \left(\frac{1}{3}\right)^x$ for $x \geq -3$

   (e) $f(x) = 6^{-x} + 5$ for $x \geq -5$ (f) $f(x) = 6^{-x} - 4$ for $x \geq 0$

   (g) $f(x) = 2 \times 5^x$ for $x \leq 6$ (h) $f(x) = 4 \times 5^x$ for $x \leq 10$

4. For each of the following functions calculate:

   (i) $f(0)$ (ii) $f(3)$

   (iii) $f(-1)$ (iv) $f\left(\frac{1}{2}\right)$

   (a) $f(x) = 5^x$ (b) $f(x) = 5^{2x}$

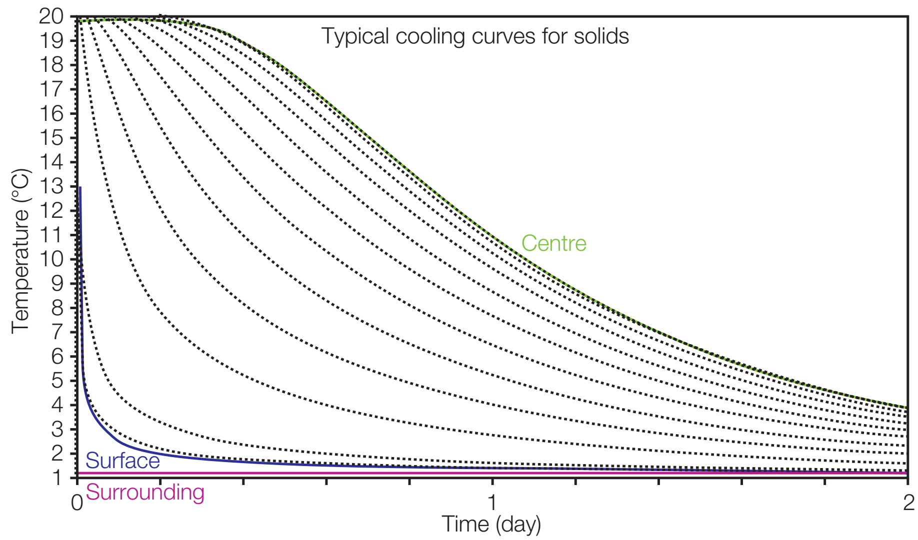
   (c) $f(x) = 5^{2x+3}$ (d) $f(x) = 5^{2x-7}$

   (e) $f(x) = 1.608 \times 5^x$ (f) $f(x) = 3.264 \times 10^{2x+4}$

   (g) $f(x) = 5.27 \times 10^{3x-8}$ (h) $f(x) = 4.376^x$

   (i) $f(x) = 34 \times 2^{1.45x}$ (j) $f(x) = 90 \times 10^{1.25x-3.578}$

19.2 Exponential models



Powered by AKTSAG.

Exponential curves can be used as efficient models for a wide range of scientific and economic phenomena, such as the spread of a new technology or the way a virus infects a population. Although you often hear the term 'exponential' used to describe any growth pattern that shows a rapid increase (or decrease), it is important to be aware that exponential models will not always fit data outside a certain range; for instance, realistically populations do not increase forever without limit. So, while exponential models are very convenient, to make the best use of them we also need to understand their limitations.

Exponential functions can be used to model both 'growth' and 'decay'; in the modelling context these phenomena are interpreted very broadly.

You already encountered a particular form of exponential function when you were studying compound interest, inflation and depreciation. For example, the formula for compound interest: $FV = PV(1+r)^n$.

RR *Look back at Chapter 4 to refresh your memory on compound interest, inflation and depreciation.*

Compare $f(x) = ka^x$ with the formula $FV = PV(1+r)^n$. If you replace PV by k, n by x, and $(1 + r)$ by a, then you can see that the compound interest formula is just an exponential function.

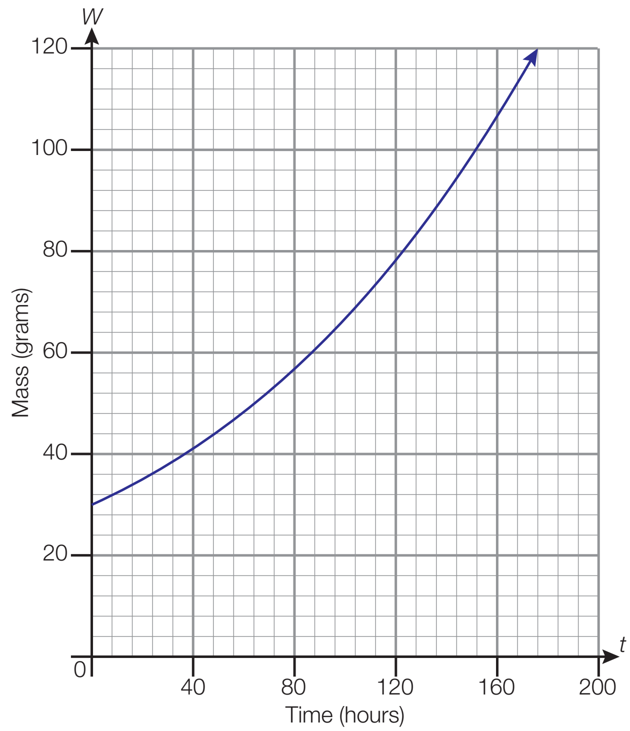
In this section we will look at some other, non-financial, models, such as cooling and the growth or decay of populations.

In a laboratory that studies bacteria, the researchers are interested in the pattern of growth or decline of the bacterial populations. Suppose Jin has 30 g of bacteria originally and has established that the colony will grow according to the function:

$$W(t) = 30 \times 1.008^t$$

where $W(t)$ represents the mass of the colony at time t, measured in hours.

She draws a graph of the growth she expects to see in the first week. Plotting the mass of the bacteria colony against time, she gets the following graph:



What is the mass of bacteria that she can expect to find at the end of the first week?

At the end of the week, $t = 7 \times 24 = 168$ hours, so the mass should be:

$$W(168) = 30 \times 1.008^{168} = 114\,\text{g}$$

At the end of the week, something contaminates Jin's sample and the bacteria start dying. There are 50 g left after five days, and Jin wants to find a function that will model the decay of the population.

Let the new function for modelling the decay of the bacterial population be called $H(t)$, which represents the mass of the colony t hours **after the contamination**. (Note that this 't' has a different meaning from the 't' in $W(t)$.)

Jin immediately writes $H(t) = 114 \times a^t$, so that $H(0) = 114$, and needs to find a value for a.

hint

The 114 in the expression for $H(t)$ comes from the mass of the colony at the end of the first week, when the contamination occurred.

She knows that $H(5) = 50$ or, in other words, $50 = 114 \times a^{5\times 24}$. So:

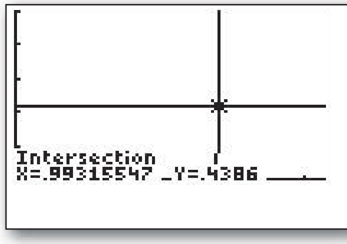
$$50 = 114 \times a^{120}$$

$$a^{120} = \frac{50}{114}$$

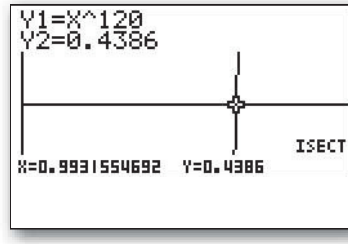
$$a^{120} = 0.4386$$

One method to find the value of a from the above equation is to draw the graph of the left-hand side of the equation on the same axes as the graph of the right-hand side of the equation, and see where they intersect. So Jin would enter $y_1 = x^{120}$ and $y_2 = 0.4386$ into her GDC and plot their graphs (see '*19.1 Solving growth and decay problems*' on page 683 of the GDC chapter for a reminder of how to solve exponential equations using your GDC):

TEXAS



CASIO



Alternatively, she could take the 120th root of 0.4396 to get the same answer:

$$a = \sqrt[120]{0.4386} = 0.993$$

Therefore, the population decay function is $H(t) = 114 \times 0.993^t$.

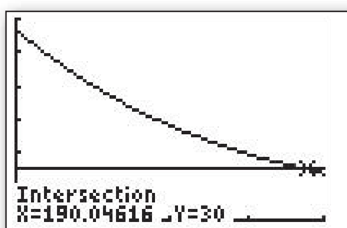
With this function, it is possible to answer questions such as 'how long will it take for the mass of the colony to decrease to the original 30 g?' All we need to do is solve the equation $H(t) = 30$ for the value of t:

$$H(t) = 30$$

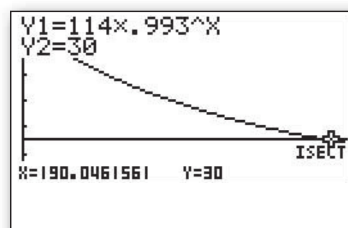
$$114 \times 0.993^t = 30$$

By drawing the graphs of $y_1 = 114 \times 0.993^x$ and $y_2 = 30$, we see that $t = 190$ hours.

TEXAS



CASIO



Worked example 19.3

Q. The population of Inverness was 12 000 in 2010. The town council estimates that the population is growing by 3% each year.

The growth can be modelled by the function $P(t) = 12000 \times 1.03^t$, where t is the number of years after 2010.

(a) Estimate the population of Inverness in 2020. Give your answer to the nearest ten people.

(b) In which year will the population be double that in 2010?

continued . . .

2020 is ten years after 2010, so take $t = 10$ and evaluate $P(10)$.

A. (a) $P(10) = 12000 \times 1.03^{10} = 16100$ (3 s.f.)

If the population doubles, that means $P(t) = 2 \times 12000$. So solve this equation for t.

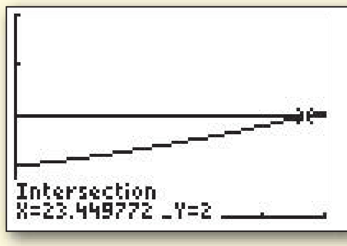
(b) $2 \times 12000 = 12000 \times 1.03^t$

Dividing through by 12000,

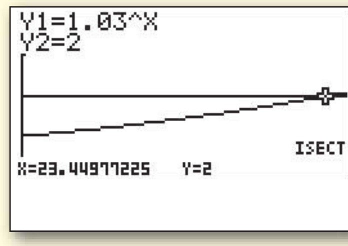
$2 = 1.03^t$

Draw the graphs of $y_1 = 1.03^x$ and $y_2 = 2$. Find their point of intersection; see section '19.1' in the GDC chapter if you need to

TEXAS



CASIO



Remember that t is not the final answer; you need to add this value to 2010.

$t = 23.4$, so the population will have doubled in 2033.

hint

The number e is a mathematical constant that often appears in growth or decay functions. It is an irrational number, e = 2.71828…, with an infinite number of decimals that do not show any repeating pattern. The concept of e is not in the syllabus for this course, but you may meet it in other contexts.

Worked example 19.4

Q. A cup of coffee is left to cool. It is estimated that its temperature decreases according to the model $T(t) = 90e^{-0.11t}$, where e is a constant with value approximately 2.718, and time t is measured in minutes.

(a) What is the initial temperature of the coffee?

(b) Calculate the temperature after 5 minutes.

(c) How long will it take for the coffee to cool to a room temperature of 21°C?

'Initial' means at $t = 0$, so the initial temperature is found by evaluating $T(0)$.

A. (a) $T(0) = 90 \times 2.718^0 = 90 \times 1 = 90°C$

Substitute $t = 5$ into the exponential function.

(b) $T(5) = 90 \times 2.718^{-0.11 \times 5} = 51.9°C$

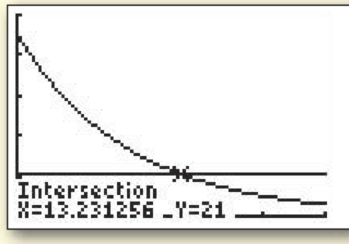
continued . . .

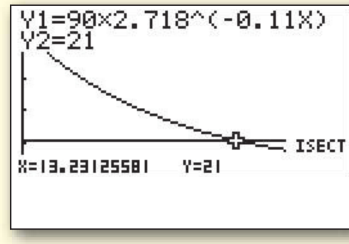
Now we want to find the value of t for which $T(t) = 21$. In other words, we need to solve this equation for t.

(c) $90 \times 2.718^{-0.11t} = 21$

On your GDC, draw the graphs of $y_1 = 90 \times 2.718^x$ and $y_2 = 21$.

TEXAS **CASIO**





Find the point of intersection.

$t = 13.2$ minutes

GDC

On the GDC you use 'x' as the variable, but when writing down your answer, remember to use the correct letter given in the question, which is 't' in Worked example 19.4.

Exercise 19.2

1. The population of a town is growing at a constant rate of 2% per year. The present population is 48 000. The population P after t years can be modelled as:

$$P = 48000 \times 1.02^t$$

(a) Calculate the population of the town:

(i) after 5 years

(ii) after 12 years

(iii) after 14.5 years.

(b) Find how long it will take for the population of the town to reach:

(i) 60 000 (ii) 100 000.

(c) Find how long it will take for the population of the town to:

(i) double (ii) treble.

2. The population of a large city was 5.5 million in the year 2000. Assume that the population of the city can be modelled by the function:

$$P(t) = 5.5 \times 1.0225^t$$

where $P(t)$ is the population in millions and t is the number of years after 2000.

(a) Calculate an estimate of the city's population in:

(i) 2010 (ii) 2018.

(b) In what year does the population of the city become double that in 2000?

The population of another city can be modelled as:

$P(t) = 6.05 \times 1.01^t$

where t is the number of years after the year 2000.

(c) In which year will the two cities have the same population?

3. Ali has been prescribed medication by his doctor. He takes a 200 mg dose of the drug. The amount of the drug in his bloodstream t hours after the initial dose is modelled by:

$C = C_0 \times 0.875^t$

(a) State the value of C_0.

(b) Find the amount of the drug in Ali's bloodstream after:

(i) 4 hours (ii) 8 hours.

(c) How long does it take for the amount of drug in the bloodstream to fall below 40 mg?

4. The population of bacteria in a culture is known to grow exponentially. The growth can be modelled by the equation:

$N(t) = 420e^{0.0375t}$ (where $e = 2.718282$)

Here $N(t)$ represents the population of the bacteria t hours after the culture was initially monitored.

(a) What was the population of the bacteria when monitoring started?

(b) Calculate $N(4)$.

(c) Work out $N(48)$.

(d) Calculate $N(72)$.

(e) How long does it take for the population of the bacteria to exceed 1000?

(f) How long does it take for the population of the bacteria to treble?

5. A liquid is heated to 100°C and then left to cool. The temperature θ after t minutes can be modelled as:

$\theta = 100 \times e^{-0.0024t}$ (where $e = 2.718282$)

(a) Calculate the temperature of the liquid after:

(i) 20 minutes (ii) 50 minutes (iii) 2 hours.

(b) How long does it take for the temperature of the liquid to drop to 80°C?

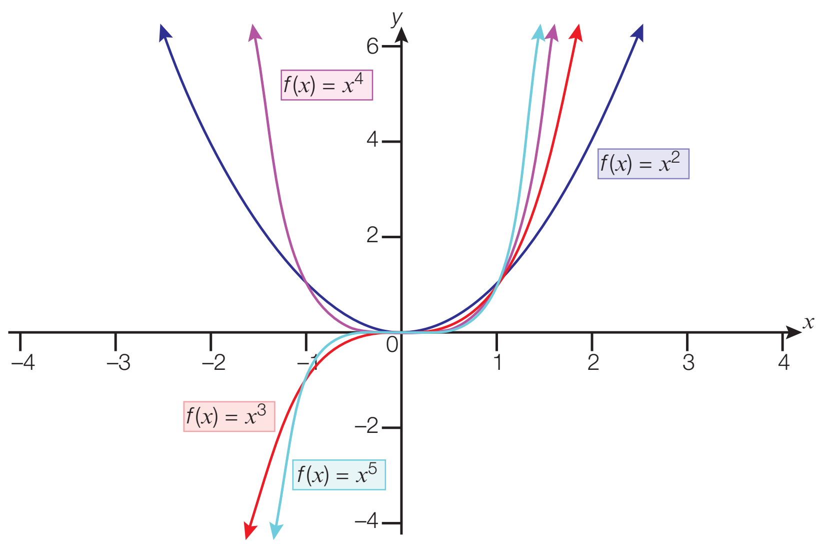
(c) Work out the time taken for the temperature of the liquid to drop to 40°C.

19.3 Polynomial functions

The function $f(x) = ax^n + bx^m + \ldots$, where the powers of x (the n and m in the expression) are **positive integers**, is called a **polynomial**.

You have already studied two types of polynomial: linear functions, where the highest power of x is 1, and quadratic functions, where the highest power of x is 2. Now you will meet some others. The **degree** of the polynomial is the highest power (or exponent, or index) that appears in the function's expression; so a linear function is of degree 1, and a quadratic function is of degree 2.

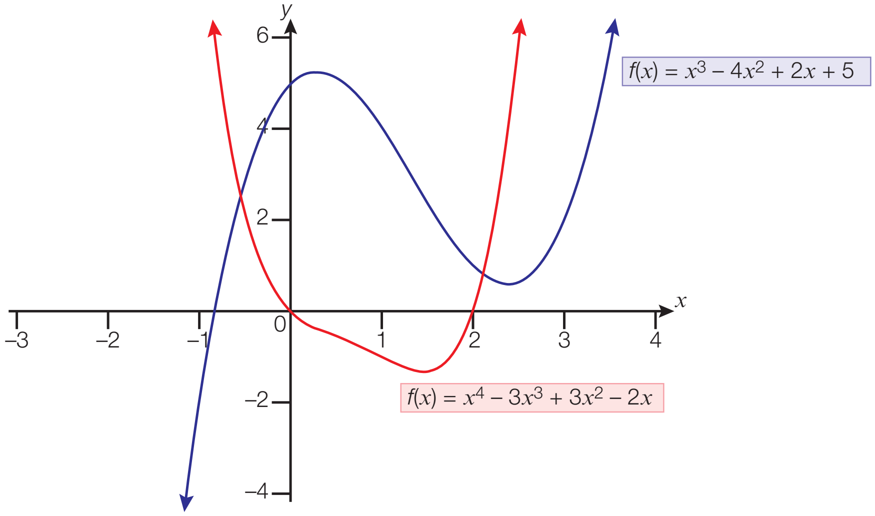
The simplest polynomial of degree n is the function $f(x) = x^n$. Draw some graphs of $f(x) = x^n$ for various values of n:



You can see that:

- The curves are all centred at the origin.
- If the power of x is even, the curve is symmetrical about the y-axis.
- If the power of x is odd, the curve has 180° rotational symmetry about (0, 0).

As you add more terms to the function, the graph becomes more interesting and develops more curves.



Name of function	General algebraic expression	Degree of polynomial	Maximum number of turning points
Linear	$f(x) = ax + b$	1	0
Quadratic	$f(x) = ax^2 + bx + c$	2	1
Cubic	$f(x) = ax^3 + bx^2 + cx + d$	3	2
Quartic	$f(x) = ax^4 + bx^3 + cx^2 + dx + e$	4	3
Quintic	$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$	5	4

For each degree of polynomial, there is a maximum number of **turning points** that the graph can have. But some polynomials of that degree will have fewer turning points; for example, not all polynomials of degree 4 will have 3 turning points. It is best to draw any given polynomial function on your GDC to get an idea of what shape to expect.

Worked example 19.5

Q. Look at the following functions and write down the degree in each case:

(a) $f(x) = 5x^4 - 6x^5 + 4x - 3$

(b) $g(x) = 3 + x - x^3 - \frac{1}{x}$

(c) $h(x) = 7x + 2$

(d) $j(x) = 1 - x - x^3 + x^2$

The term with the highest power is $-6x^5$.

A. (a) The degree is 5.

continued . . .

This has a rational term $\frac{1}{x}$ (which is a negative power of x).

Remember that in a polynomial all exponents of x should be positive integers.

(b) It is not a polynomial.

The term with the highest power is $7x$; this is a linear function.

(c) The degree is 1.

The term with the highest power is $-x^3$.

(d) The degree is 3.

As polynomial functions become more complicated, they may also have more intercepts on the x-axis. Finding these intercepts is very similar to finding the solutions (or roots) of a quadratic equation, and you can use your GDC to do this. (See '*19.3 Solving polynomial equations (a) using a graph*' and '*(b) using an equation solver*' on page 685 of the GDC chapter if you need a reminder.)

See section '*19.3 Solving polynomial equations*' on page 685 of the GDC chapter if you need to.

Worked example 19.6

Q. Sketch the following curves with your GDC. In each case write down the y-intercept and use your GDC to find any x-intercepts.

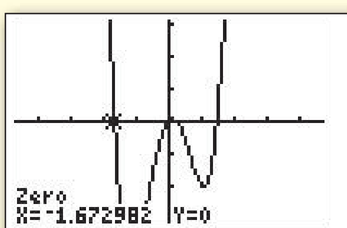
(a) $f(x) = 2x^4 - 5x^2 + x$

(b) $g(x) = 2 - 5x^2 + 2x^3$

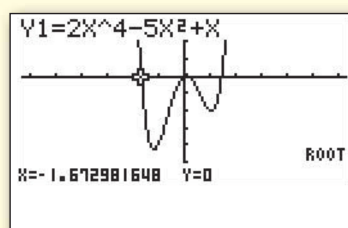
(c) $h(x) = x^4 + x^3 - 6$

A. (a)

TEXAS



CASIO



For the y-intercept, look at the graph or evaluate $f(0)$.

If you zoom in near the origin, you should see that there are two x-intercepts there; be careful not to miss one.

y-intercept: $(0, 0)$

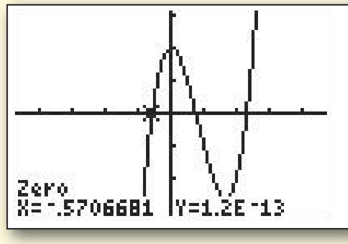
x-intercepts: $(-1.67, 0)$, $(0, 0)$, $(0.203, 0)$, $(1.47, 0)$

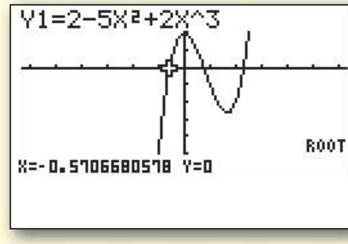
continued . . .

(b)

TEXAS

CASIO





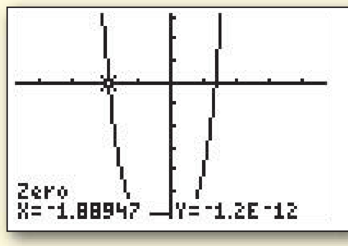
y-intercept: (0, 2)

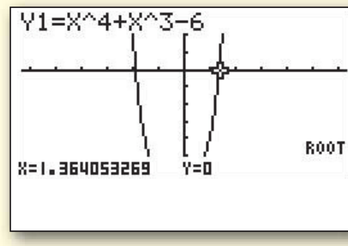
x-intercepts: (−0.571, 0), (0.758, 0), (2.31, 0)

(c)

TEXAS

CASIO





y-intercept: (0, −6)

x-intercepts: (−1.89, 0), (1.36, 0)

Remember that in a polynomial function the powers of x should all be positive integers. If a function of the form $f(x) = ax^n + bx^m + \dots$ contains one or more powers of x that are **negative** integers, then $f(x)$ is not a polynomial. Instead, it is a **rational function**. For example, $f(x) = 5x^2 + \frac{4}{x}$ contains a negative power of x (because $\frac{4}{x} = 4x^{-1}$) and can be written as $f(x) = \frac{5x^3+4}{x}$, which is a ratio of two polynomials. The graphs of such rational functions will have $x = 0$ (the y-axis) as a vertical asymptote.

RR *Rational functions and their graphs were covered in section 17.3.*

Exercise 19.3

1. Sketch the following curves on your GDC. In each case:

(i) Write down the y-intercept. (ii) Find any x-intercepts.

(a) $f(x) = x^3 - 7x + 6$

(b) $f(x) = x^3 + x^2 - 5x + 3$

(c) $f(x) = 8 - 2x - 5x^2 - x^3$

(d) $f(x) = 6 - 7x - 18x^2 - 5x^3$

(e) $f(x) = x^3 + 3x^2 - 4$

(f) $f(x) = x^3 + 5x^2 - 6x$

(g) $f(x) = 2x^3 + x^2 - 13x + 6$

(h) $g(x) = x^4 - 6x^2 - 8$

(i) $g(x) = 2x^4 + 7x^3 - 6x^2 - 7x + 4$

(j) $g(x) = x^5 - 9x^3 + x^2 - 9$

RR You learned about the domain and range in Chapter 17.

2. Sketch the following curves on your GDC for the stated domains. In each case:

(i) write down the coordinates of the intercepts with the axes

(ii) state the range of the function.

(a) $f(x) = x^2 - x - 12$ for $-4 \le x \le 5$

(b) $f(x) = 1 + 7x - x^2$ for $-1 \le x \le 7$

(c) $f(x) = 8x + 2x^2 - x^3$ for $-4 \le x \le 6$

(d) $f(x) = x^3 - 2x^2 - 5x + 6$ for $-2 \le x \le 4$

(e) $g(x) = 3 + 7x^3 - 6x^2 - 2x^4$ for $-1 \le x \le 3$

(f) $g(x) = x^3 + x^2 - 5x + 4$ for $-4 \le x \le 6$

(g) $f(x) = \frac{1}{4}x^3 - 7x^2 - \frac{1}{3}x + 10$ for $-2 \le x \le 2$

(h) $f(x) = x^4 - \frac{1}{2}x^3 + x^2 - x + 6$ for $0 \le x \le 2$

19.4 Modelling with polynomial functions

Some practical situations can be described fairly well by a polynomial model. As with other models, the first step is to collect data and plot it on a graph. Then try to find a curve that fits the data and helps you to study the problem in more depth. The function represented by the curve can be used to estimate values that were not collected as part of the original data set. You can also work out when that function will reach certain values of interest.

Worked example 19.7

Q. Bram is working in a laboratory, measuring the speed (in metres per second) at which a particle drops through a certain liquid. He draws a graph of velocity against time, and thinks that it will fit a cubic model. After a bit more work, he decides that the equation $v(t) = 34.3t^3 - 25.0t^2 + 6.28t + 0.16$ would describe the data well during the first second.

(a) Use this equation to estimate the velocity of the particle when $t = 0.5$ seconds.

(b) Calculate the time at which the velocity of the particle is $0.6\,\text{m s}^{-1}$.

Finding the velocity at $t = 0.5$ means to replace t by 0.5 in the equation for $v(t)$, i.e. to evaluate $v(0.5)$.

A. (a) $v(0.5) = 34.3(0.5)^3 - 25.0(0.5)^2 + 6.28(0.5) + 0.16 = 1.34\,\text{m s}^{-1}$

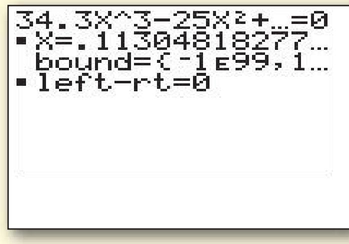
continued . . .

We want to find the value of t for which $v(t) = 0.6$. This involves solving a polynomial equation.

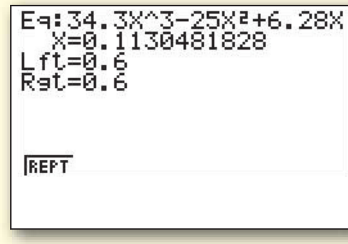
(b) $34.3t^3 - 25.0t^2 + 6.28t + 0.160 = 0.6$

You can use the equation solver on your GDC (or you could plot graphs and find the point of intersection). See '*19.3 (b) Solving polynomial equations using an equation solver*' on page 685 of the GDC chapter if you need to.

TEXAS



CASIO



The velocity of the particle is 0.6 m s^{-1} after 0.113 seconds.

Worked example 19.8

Q. Risha has been monitoring the level of water in a local reservoir over the past three months. She finds that the depth of the water can be modelled by the function:

$$D(t) = 0.024t^3 - 0.87t^2 + 4.9t + 39$$

where t is measured in weeks.

(a) Find the depth of water in the reservoir when Risha begins her study.

(b) After how many weeks is the depth of water in the reservoir at 25 m?

(c) During Risha's study, the water level first rises and then falls. What is the maximum depth of water that Risha measures?

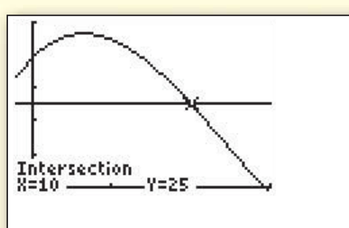
The beginning of the study corresponds to $t = 0$, so evaluate $D(0)$.

A. (a) $D(0) = 39$ m

We want to find the value of t for which $D(t) = 25$. This involves solving a cubic equation, which you can do by drawing graphs on your GDC and finding the points of intersection (or using the equation solver program). See '*19.3 Solving polynomial equations*' on page 685 of the GDC chapter if you need to.

(b) $0.024t^3 - 0.87t^2 + 4.9t + 39 = 25$

TEXAS



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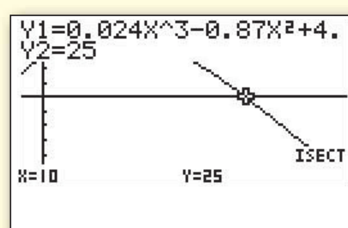


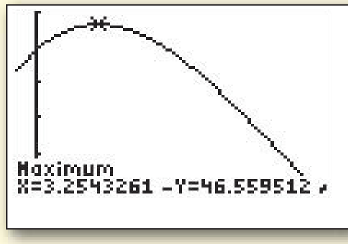
From GDC: $t = 10$ weeks

Draw the graph of $D(t)$ and find the coordinates of the maximum point on the curve. (See '*18.1 Using a graph to find the vertex and line of symmetry of a parabola*' on page 680 of the GDC chapter for a reminder of how to find the minimum and maximum points, if you need to.)

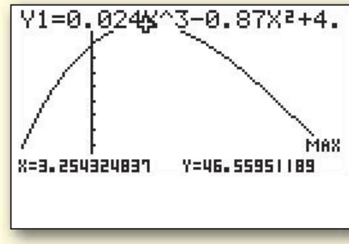
continued . . .

(c)

TEXAS



CASIO



The maximum depth is 46.6 m, which occurs 3.25 weeks after Risha starts her study.

Exercise 19.4

1. Jana has been studying trends in the exchange rate between the US dollar and the euro over the 12 months in 2011. She has suggested an approximate cubic model; the equation of the modelling function is:

$$f(x) = 0.0003x^3 - 0.0088x^2 + 0.0748x + 1.2582$$

where $f(x)$ is the number of US dollars per euro, x months after 1 January 2011.

(a) Using this model, estimate the exchange rate of USD to EUR on:

(i) 1 April 2011 (ii) 1 August 2011 (iii) 1 November 2011.

(b) Use your GDC to sketch the graph of $f(x)$, and hence estimate the peak value of the exchange rate over the 12-month period.

2. The total population of the world in billions between 1950 and 2010 can be modelled by the following function:

$$p(x) = 2.557 + 0.3554x + 0.145x^2 - 0.0207x^3 + 0.001316x^4 - 0.00003718x^5$$

where $p(x)$ is the mid-year population of the world in billions, x decades after 1950.

(a) The actual mid-year populations are given in the following table. Use the model to calculate the estimated populations. Work out the percentage error in using the model to estimate the population of the world.

Decade		1970	1990	2010
Population (billions)	**Actual**	3.706618	5.278640	6.848933
	Estimated from model			
Percentage error				

(b) Use the model to estimate the projected mid-year population of the world in:

(i) 2020 (ii) 2030 (iii) 2040.

3. Marko has studied trends in the price of silver on the commodities market over a six-year period. He has suggested a model for the price per ounce of silver, in US dollars, over the six years since 2000.

According to Marko, the price of silver can be modelled by the function

$$f(x) = -0.0669x^5 + 1.0383x^4 - 5.9871x^3 + 16.817x^2 - 22.471x + 15.78$$

where $f(x)$ is the price in USD per ounce of silver and x is the number of years after 1 January 2000.

(a) Use Marko's model to complete the following table:

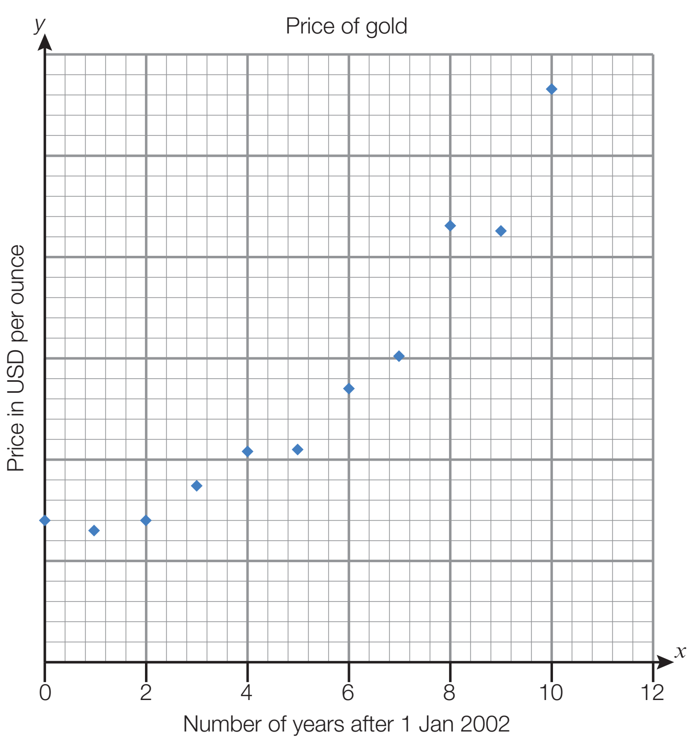
Year	2000	2002	2004	2006
Price of silver on 1 January				

(b) Use Marko's model to predict the price of silver on 1 January 2005.

The actual price of silver on the commodities market on 1 January 2004 was US$9.08 per ounce.

(c) Calculate the percentage error in using Marko's model to determine the price of silver on 1 January 2004.

4. Martha found the following diagram in an Economics journal. The diagram shows the price of gold per ounce, in US dollars, over a ten-year period. However, the numbers on one of the axes are missing.



After some effort, Martha managed to find an approximate function $g(x)$ to fit the trend of prices. Her function $g(x)$ is:

$$g(x) = -0.007x^5 + 0.2527x^4 - 2.7818x^3 + 20.306x^2 - 23.773x + 278$$

where $g(x)$ is the price in USD per ounce of gold x years after 1 January 2002.

(a) Use Martha's model to complete the following table:

Year	2002	2004	2006	2008	2010	2012
Price of gold on 1 January						

(b) Use Martha's model to estimate the price of gold on 1 January 2011.

(c) If the actual price of gold on 1 January 2011 was US$853.60 per ounce, calculate the percentage error in using Martha's model to estimate the price on 1 January 2011.

Summary

You should know:

- what an exponential function is, and how to recognise one and its graph:

  $f(x) = ka^x + c$

  or $f(x) = ka^{-x} + c$

  (where $a \in \mathbb{Q}^+$, $a \neq 1$, $k \neq 0$)

RR ***Recall from Chapter 1 that $\mathbb{Q}$ denotes 'rational numbers'; the superscript '+' means all positive rational numbers.***

- how to find the horizontal asymptote of an exponential function
- how to use exponential models
- what a polynomial function is, and how to recognise one and its graph:

  $f(x) = ax^n + bx^m + \ldots$ (where the powers of x are positive integers)

- how to use polynomial functions as mathematical models.

Mixed examination practice

Exam-style questions

1. A function is defined as $g(x) = -2x^4 + 5x^3 + 6x^2 - 7x - 4$. Sketch the graph of the function on your GDC, with the domain $-2 \le x \le 4$, and use it to answer the following questions.

(a) State the coordinates of the y-intercept.

(b) Find all intercepts with the x-axis.

(c) Determine the range of the function.

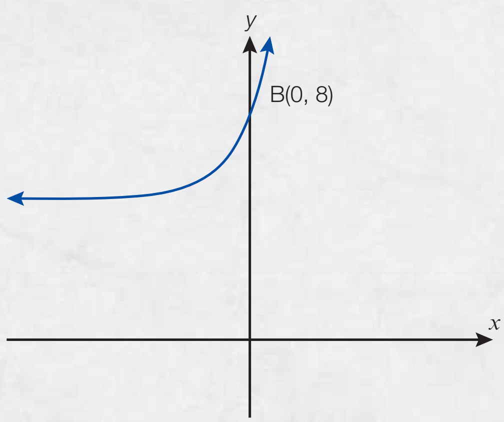
2. Use your GDC to draw the graph of the function $f(x) = 8^{-x}$.

(a) State the coordinates of the y-intercept.

(b) Write down the equation of the asymptote.

(c) State the range of the function.

3. The curve with equation $y = A \times 2^x + 5$ passes through the point B with coordinates (0, 8).



(a) Find the value of A.

(b) Write down the equation of the asymptote.

4. The temperature of a bowl of soup left to cool outside can be modelled by an exponential function. The temperature θ in degrees Celsius after t hours is:

$\theta = 24 + 56e^{-1.754t}$ (where e = 2.718282)

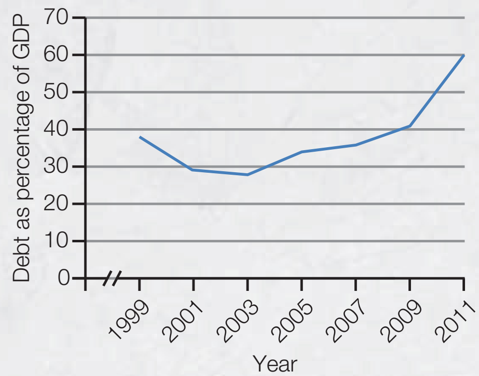
(a) Calculate the temperature of the soup after:

(i) half an hour (ii) 48 minutes (iii) two hours (iv) 156 minutes.

(b) Find how long it takes for the temperature of the soup to drop to:

(i) 70°C (ii) 34°C (iii) 28°C.

5. The diagram below illustrates the trend in the debt of a major western country as a percentage of its GDP from 1999 to 2011.



The curve can be modelled approximately by a polynomial with equation:

$$p(x) = 0.017045x^4 - 0.402146x^3 + 3.38636x^2 - 10.37626x + 38.2121$$

where $p(x)$ is the national debt as a percentage of GDP and x is the number of years after 1999.

Use the model to complete the following table.

Year	2000	2002	2004	2006	2008	2010
Debt (% of GDP)						

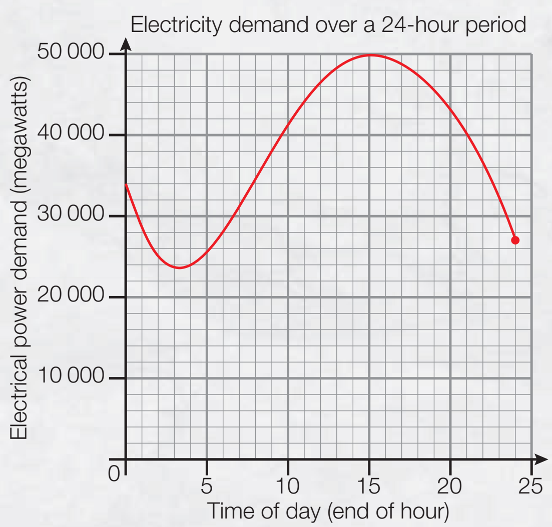
6. After a person took a dose of a drug, the initial concentration of the drug in their bloodstream was 8 mg/ml. Six hours later, the concentration of the drug in the bloodstream had dropped to 4.8 mg/ml.

Suppose that the concentration C, in mg/ml, of the drug in the bloodstream t hours after the initial dose can be modelled by:

$$C = C_0 \times e^{kt}$$

(you may assume that e = 2.718282). Find the values of C_0 and k.

7. Pauli is modelling the electricity demand curve of a major country on a typical day. He has fitted a curve to approximate the original demand curve:



The equation of his polynomial modelling function is:

$$p(x) = -0.0037x^6 + 0.2351x^5 - 4.177x^4 - 22.112x^3 + 1178.7x^2 - 6637.9x + 33969$$

where $p(x)$ is the demand in megawatts x hours after midnight.

(a) Use Pauli's model to estimate the electricity demand at:

(i) 6.00 a.m. (ii) 12 noon (iii) 6.00 p.m. (iv) 9.00 p.m.

Give your answers to the nearest 10 megawatts.

(b) Use your GDC to find the times of the day when electricity demand is estimated to be:

(i) 38 000 megawatts (ii) 43 000 megawatts.

Past paper questions

1. (a) Sketch the graph of the function $f(x) = \dfrac{2x+3}{x+4}$ for $-10 \le x \le 10$, indicating clearly the axis intercepts and any asymptotes. *[6 marks]*

(b) Write down the equation of the vertical asymptote. *[2 marks]*

(c) On the same diagram sketch the graph of $g(x) = x + 0.5$. *[2 marks]*

(d) Using your graphical display calculator write down the coordinates of **one** of the points of intersection on the graphs of f and g, **giving your answer correct to five decimal places**. *[3 marks]*

(e) Write down the gradient of the line $g(x) = x + 0.5$. *[1 mark]*

(f) The line L passes through the point with coordinates $(-2, -3)$ and is perpendicular to the line $g(x)$. Find the equation of L. *[3 marks]*

[Total 17 marks]

[May 2008, Paper 2, TZ1, Question 1] (*© IB Organization 2008*)

2. In an experiment it is found that a culture of bacteria triples in number every four hours.

There are 200 bacteria at the start of the experiment.

Hours	0	4	8	12	16
No. of bacteria	200	600	a	5400	16200

(a) Find the value of a. *[1 mark]*

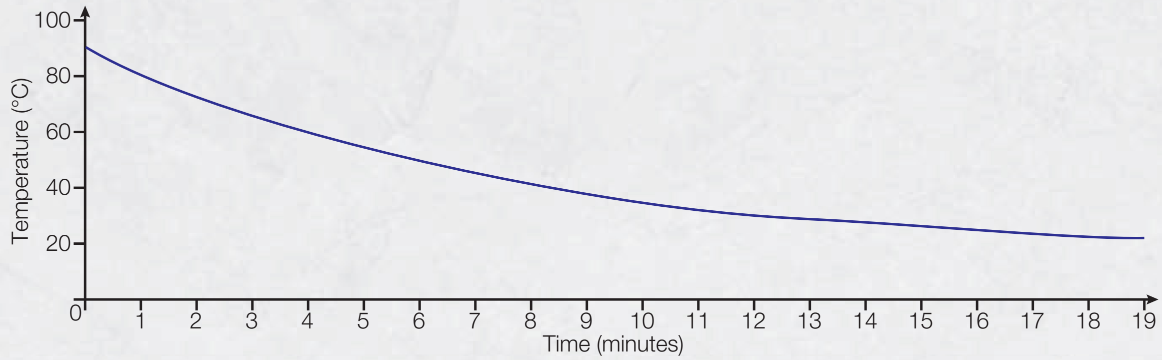
(b) Calculate how many bacteria there will be after one day. *[2 marks]*

(c) Find how long it will take for there to be two million bacteria. *[3 marks]*

[Total 6 marks]

[May 2008, Paper 1, TZ1, Question 15] (*© IB Organization 2008*)

3. The following graph shows the temperature in degrees Celsius of Robert's cup of coffee, t minutes after pouring it out. The equation of the cooling graph is $f(t) = 16 + 74 \times 2.8^{-0.2t}$ where $f(t)$ is the temperature and t is the time in minutes after pouring the coffee out.



(a) Find the initial temperature of the coffee. *[1 mark]*

(b) Write down the equation of the horizontal asymptote. *[1 mark]*

(c) Find the room temperature. *[1 mark]*

(d) Find the temperature of the coffee after 10 minutes. *[1 mark]*

If the coffee is not hot enough it is reheated in a microwave oven. The liquid increases in temperature according to the formula

$$T = A \times 2^{1.5t}$$

where T is the final temperature of the liquid, A is the initial temperature of coffee in the microwave and t is the time in minutes after switching the microwave on.

(e) Find the temperature of Robert's coffee after being heated in the microwave for 30 **seconds** after it has reached the temperature in part (d). *[3 marks]*

(f) Calculate the length of time it would take a similar cup of coffee, initially at 20°C, to be heated in the microwave to reach 100°C. *[4 marks]*

[Total 11 marks]

[Nov 2007, Paper 2, TZ0, Question 3(i)] (© *IB Organization 2007*)