7 Introduction to differential calculus

Surfing and snowboarding are among the many sports that use a flat board in contact with a curved surface. The skill lies in how the surfer or snowboarder manipulates the board against the curve to effect changes in velocity and acceleration.

Velocity and acceleration are variables that are described in terms of the rate of change of one physical quantity in relation to another; velocity is the rate of change of displacement with respect to time, and acceleration is the rate of change of velocity with respect to time.

Calculus is the mathematics of change — the study of quantities that do not stay the same and whose rate of change is important in a particular context. It is used in fields as varied as physics and finance, architecture and engineering, the setting of credit card payments and the science behind computer games.

The development of calculus is one of the major achievements of mathematics. It stimulated the flowering of mathematics and science that sparked the industrial revolution and led to the growth of the technology that we know today.

Prior learning topics

It will be easier to study this topic if you have completed:

- Chapter 2
- Chapter 14
- Topic 6 (Chapters 17–19)

Chapter 20 Introduction to differential calculus

In this chapter you will learn:

- the two concepts of the derivative
  - as a rate of change of a function
  - as a gradient of a graph
- about the concept of the gradient of a curve as the gradient of a tangent line
- that if $f(x) = ax^n$, then $f'(x) = nx^{n-1}$
- how to find the derivative of functions of the form $f(x) = ax^n + bx^m + \ldots$ where all exponents are integers
- how to find the gradient of a curve at given values of x
- how to find the values of x at which a curve has a given gradient
- how to find the equation of the tangent at a given point on a curve
- how to find the equation of the line perpendicular to the tangent at a given point (the normal).

In the seventeenth century, two historical figures approached the theory of differential calculus from two very different directions. The English scientist Sir Isaac Newton viewed differential calculus in terms of physics, as the rate of change of a quantity over time. The German mathematician Gottfried Leibniz viewed differential calculus in terms of geometry, as the way in which the gradient of a curve changes over distance.

Newton developed his ideas before Leibniz, but Leibniz published his results earlier; the two men had a long feud over plagiarism. Today, both Newton and Leibniz are credited with the modern development of calculus, and it is generally accepted that they worked independently from different directions and that there was no evidence of plagiarism.

Isaac Newton (1642–1727) was an English scientist and mathematician, who worked in Cambridge and became President of the Royal Society. He made influential contributions to optics and mechanics, and his work on the paths of planets led to his formulation of calculus. He also pursued many other interests, ranging from alchemy to the study of religion. His final post was as Master of the Royal Mint in London, where he introduced the idea of a milled edge to coins.

Gottfried Leibniz (1646–1716) was born in Germany. He worked as a lawyer and librarian, but is described as a polymath — a universal genius who contributed ideas to a wide range of fields. He also travelled widely, and is said to have corresponded with more than 600 people, including many influential mathematicians and scientists of the day. Leibniz invented an early calculating machine, and worked on the mathematics of zero and one; this is now called binary mathematics and forms the basis of modern computer systems.

Mathematical concepts can take centuries to grow and yet be based on fundamentally the same idea. Archimedes proposed finding the area of a circle by looking at polygons with more and more sides; Johannes Kepler found a formula for the volume of barrels by splitting them into thinner and thinner slices. The same idea (of dividing into more and more smaller and smaller pieces) lies behind the development of 'integral calculus'. Two hundred years before Newton and Leibniz, Madhava of Sangamagrama (in present day Kerala, India) worked on the idea of infinite series and found a value for π that was accurate to 13 decimal places. Owing to his work on the links between finite algebra and infinite series, he is now also considered to be an important figure in the history of calculus.

How is it possible to reach the same conclusion from different directions? Can the development of mathematics be thought of as a straight line, or is it more like a tree diagram? (You learned about tree diagrams in Chapter 10.)

20.1 The derivative

A straight line has constant **gradient**. For a straight line plotted on (x, y) axes, the gradient is defined as:

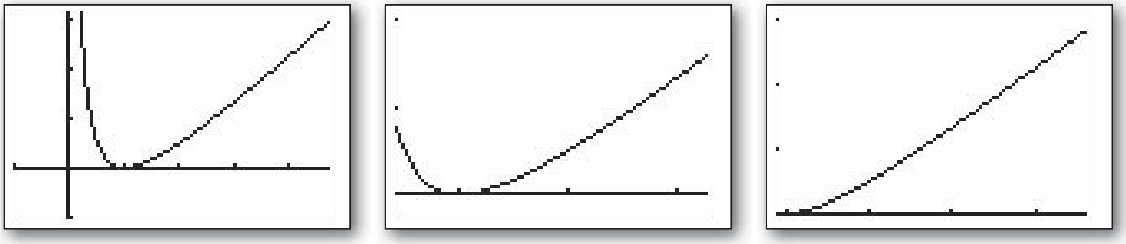
$$m = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}} = \frac{\text{change in } y}{\text{change in } x}$$

RR *You met this formula in Chapter 14.*

So a constant gradient means that the rate at which y changes with x is always the same.

How can we adapt this idea to define the gradient of a **curve**?

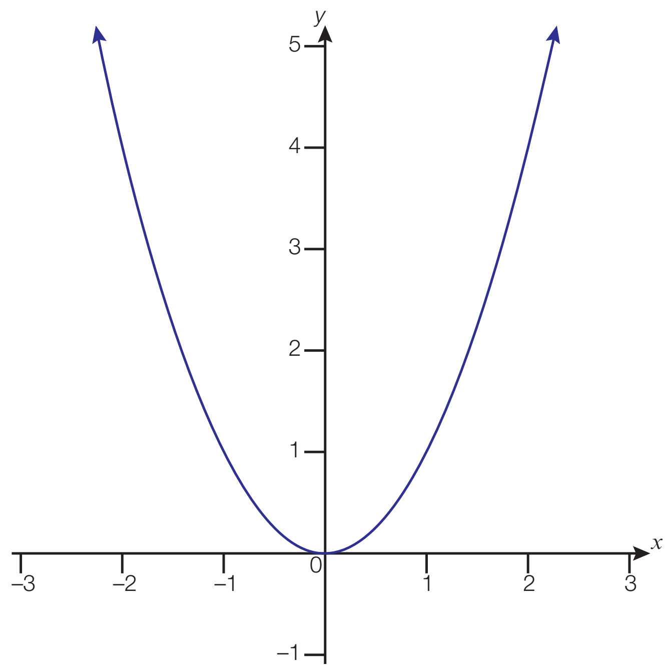
Using a GDC or maths software on a computer, you can graph a curve and 'zoom in' on it to look more closely at a small section, as shown in the following three diagrams. The more you zoom in, the more the curve looks like a straight line.



If we use the definition of the gradient of a straight line given above for a very short section of curve, we can get a good **approximation** of the 'gradient' of the curve in that region. By taking shorter and shorter sections around a particular point on the curve and calculating the gradient over them, you can obtain better and better estimates of the **gradient of the curve at that point**.

Finding the gradient of a curve at a point

Look at the graph of $y = x^2$:



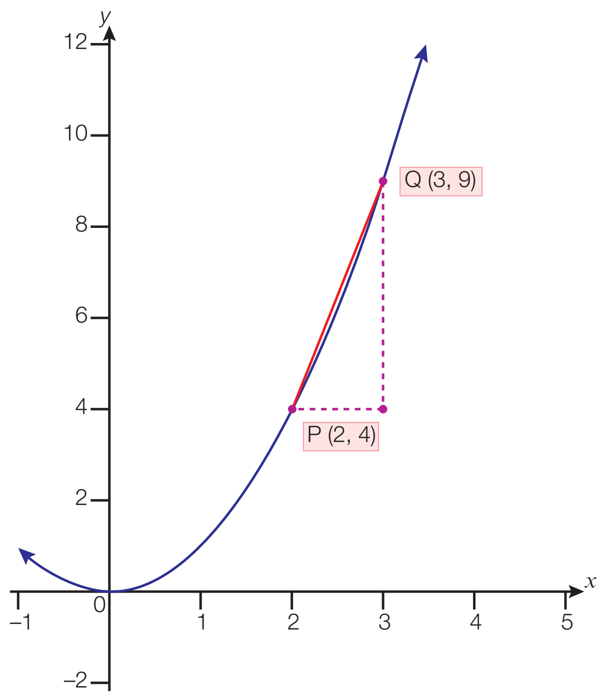
The gradient is different at every point on the curve. For example:

- when $x = -2.5$, the curve is decreasing steeply — the gradient is negative and large
- as x increases towards zero, the gradient remains negative but becomes smaller in magnitude (the curve becomes less steep)
- the curve is 'flat' at (0, 0), which means here the gradient is zero
- as x increases from 0, the curve slopes upward and gets steeper — the gradient is positive and getting larger.

The rate of change of y against x is different at every point on the curve.

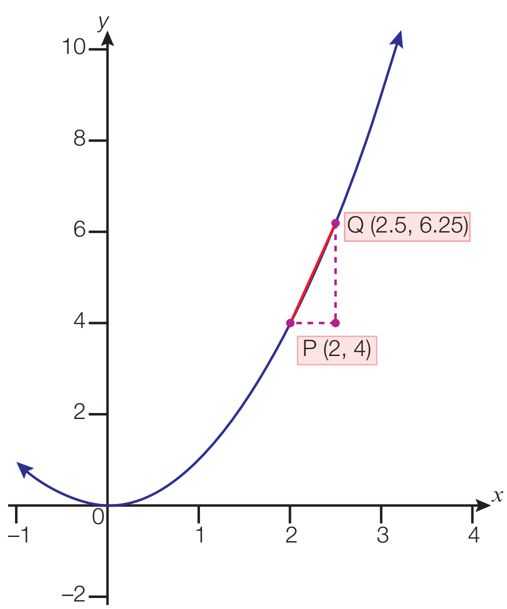
To find the gradient at any particular point P on the curve $y = x^2$, start by considering a **chord** PQ across a segment of the curve.

For example, to find the gradient at (2, 4), plot the points P(2, 4) and Q(3, 9) on the curve and join them with a straight line.



The gradient of the chord PQ is $\frac{9-4}{3-2} = 5$.

Now move Q closer to P. The chord PQ becomes shorter and also lies closer to the section of curve between P and Q. For instance, taking Q to be (2.5, 6.25):



This time, the gradient of PQ is $\frac{6.25-4}{2.5-2}=4.5$.

If we continue to push Q towards P, the values of the gradient of PQ are as follows:

P	Q	Gradient of PQ
2	3	$\dfrac{9-4}{3-2}=5$
2	2.5	$\dfrac{6.25-4}{2.5-2}=4.5$
2	2.25	$\dfrac{5.0625-4}{2.25-2}=4.25$
2	2.1	$\dfrac{4.41-4}{2.1-2}=4.1$
2	2.01	$\dfrac{4.0401-4}{2.01-2}=4.01$

You can see that as Q moves closer to P, the gradient of the line PQ gets closer to 4.

When P and Q are so close that they are effectively the same point, the line PQ becomes the **tangent** to the curve at the point (2, 4).

The tangent to a curve is a straight line that touches the curve at one single point. The gradient of the curve at a point will be the same as the gradient of the tangent at that point.

We can use the same technique as above to find the gradient of the $y=x^2$ curve at other points.

If you repeat the calculations you did for the point (2, 4) at different points on the curve, you will get results like the following:

P	Gradient of curve at P	Pattern observed
$(-3, 9)$	-6	$-3\times 2=-6$
$(-2, 4)$	-4	$-2\times 2=-4$
$(-1, 1)$	-2	$-1\times 2=-2$
$(0, 0)$	0	$0\times 2=0$
$(2, 4)$	4	$2\times 2=4$
$(4, 16)$	8	$4\times 2=8$
(x, x^2)	$2x$	$x\times 2=2x$

These results suggest that the gradient of $y=x^2$ at any point on the curve can be calculated by multiplying the x-coordinate of that point by 2. Note that the gradient of a curve depends on the position at which you are calculating it; in other words, the gradient is itself a function of x. It is often referred to as the **gradient function** of a curve.

RR ***You learned about functions in Chapter 17.***

Using more compact notation makes it easier to list results and conclusions. There are two types of notation that are commonly used. You should be familiar with both as you might meet either of them in the examinations.

The notation introduced by Leibniz is generally considered more convenient than that formulated by Newton, and is often the one that is used by teachers when they first introduce students to calculus.

	Leibniz notation	**Newton's notation**
Equation of curve or function	$y = x^2$	$f(x) = x^2$
Equation for the gradient	$\frac{dy}{dx} = 2x$	$f'(x) = 2x$

hint

Some people remember Leibniz notation as $\frac{dy}{dx} = \frac{\text{difference in } y}{\text{difference in } x}$

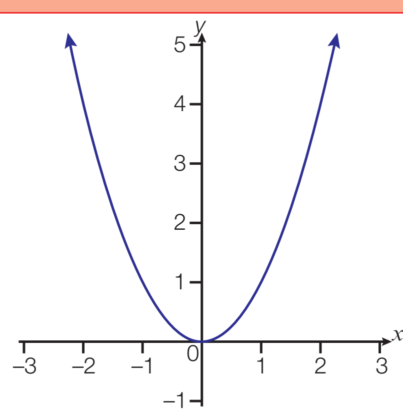
$\frac{dy}{dx}$ is pronounced 'DY by DX'.

In applications, often the independent variable is called t (for time) instead of x, in which case we would write, for example:

- in Leibniz notation, if $y = t^2$ then $\frac{dy}{dt} = 2t$
- in Newton's notation, if $f(t) = t^2$ then $f'(t) = 2t$.

20.2 Differentiation

The process of finding the gradient function of a curve is called **differentiation**. To 'differentiate' a function or the equation of a curve means to find its derivative or gradient. Both numerical differentiation and differentiation from first principles will give the following results for these curves. (See Learning links 20A on page 581 if you are interested in differentiation from first principles.)

Function $y = f(x)$	**Derivative** $\frac{dy}{dx}$	**Graph of function**
$y = x^2$	$2x$	

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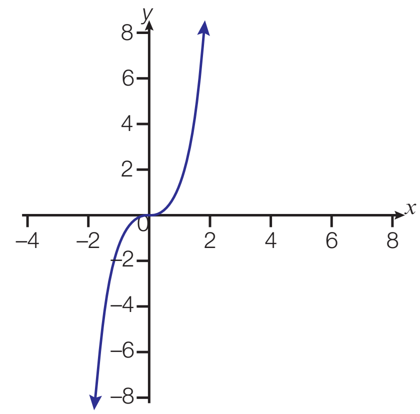
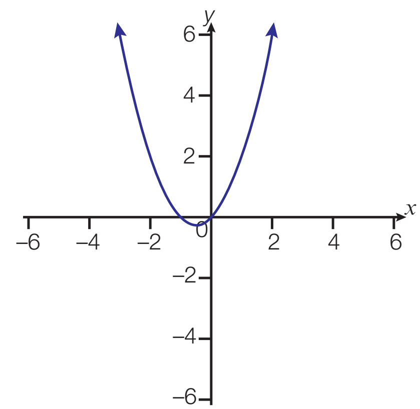
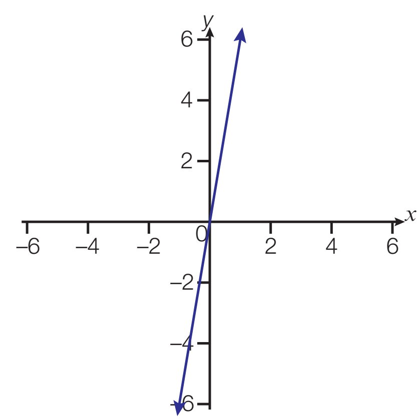
Function $y = f(x)$	Derivative $\frac{dy}{dx}$	Graph of function
$y = x^3$	$3x^2$	
$y = x^2 + x$	$2x + 1$	
$y = 6x$	6	

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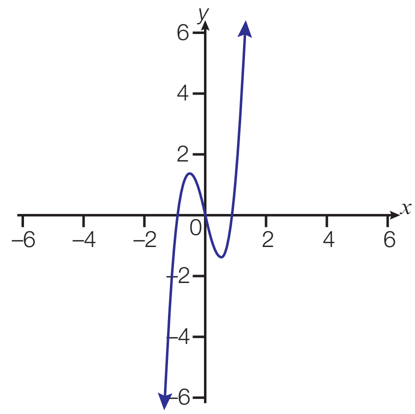
Function $y = f(x)$	**Derivative** $\frac{dy}{dx}$	**Graph of function**
$y = 5x^3 - 2x$	$15x^2 - 2$	

In each case:

- $y = f(x)$ gives the rule for plotting the original curve
- $\frac{dy}{dx}$ gives the formula for finding the gradient at any point on the curve.

If you look carefully at the results in the table above, you can see that there is a general rule relating the formula for the function and the formula for the gradient.

- If $y = x^n$, then $\frac{dy}{dx} = n \times x^{n-1}$ or, equivalently, $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ in Newton's notation
- If $y = a \times x^n$, then $\frac{dy}{dx} = n \times a \times x^{n-1} = nax^{n-1}$ or, equivalently, $f(x) = ax^n \Rightarrow f'(x) = nax^{n-1}$

The formula that you are given in the Formula booklet is in Newton's notation:

$a = \pi r^2$ $f(x) = ax^n \Rightarrow f'(x) = nax^{n-1}$

In words we say: 'to differentiate a power of x, multiply by the power and then reduce the power by one. If there is a coefficient (or constant factor), multiply the coefficient by the power'.

The following example uses both styles of notation.

Worked example 20.1

Differentiate:

(a) $y = x^4$

(b) $f(x) = 3x^2$

(c) $y = 6x^3 + 2x^2$

(d) $f(x) = \frac{1}{2}x^3 - 6x$

The power is 4, so multiply by 4 and then reduce the power by 1.

(a) $\frac{dy}{dx} = 4 \times x^{4-1} = 4x^3$

The power is 2, and there is also a coefficient 3, so multiply the coefficient by 2 and reduce the power.

(b) $f'(x) = 2 \times 3x^{2-1} = 6x^1 = 6x$

Now we have a sum of two powers of x, both with coefficients. Differentiate each term separately; first differentiate $6x^3$ and then $2x^2$. Then add the results together.

(c) $\frac{dy}{dx} = 3 \times 6x^{3-1} + 2 \times 2x^{2-1}$

$= 18x^2 + 4x$

Follow the same procedure as in (c). Remember that $x = x^1$ and reducing the power by 1 gives $x^0 = 1$.

(d) $f'(x) = 3 \times \frac{1}{2}x^{3-1} - 1 \times 6x^{1-1}$

$= \frac{3}{2}x^2 - 6x^0$

$= 1.5x^2 - 6$

hint

Remember that each term of an equation is separated by either a '+' or '–' operator or the '=' sign. The '×' and '÷' operators do not separate terms; they form part of the term.

In parts (c) and (d) of Worked example 20.1, each term has been differentiated separately.

This is the procedure to follow for all curves whose equations are made up of more than one term:

If $y = ax^n + bx^m + \ldots$, then $\frac{dy}{dx} = anx^{n-1} + bmx^{m-1} + \ldots$

This is called the 'derivative of a sum' in the Formula booklet:

$a = \pi r^2$ $f(x) = ax^n, g(x) = bx^m \Rightarrow f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$ (in Newton's notation)

Exercise 20.1

1. Find $\frac{dy}{dx}$ for each of the following functions.

(a) $y = x^5$

(b) $y = 6x^3$

(c) $y = -7x^4$

(d) $y = \frac{4}{3}x^6$

(e) $y = x^2 - 4x$

(f) $y = 8x^9 - 15x^2$

(g) $y = 19x + 11x^2$

(h) $y = 3x^2 + 5x - 7x^3$

2. Find $f'(x)$ for each of the following functions.

(a) $f(x) = x - x^7$

(b) $f(x) = 20x^2 - x^9$

(c) $f(x) = 11x^3 - 9x^2 - 7x$

(d) $f(x) = 6x^5 - x^3 - 13x$

(e) $f(x) = 10x - 9x^2 - x^4$

(f) $f(x) = 8x^4 - 5x^3 + 2x^2 + 7x$

(g) $f(x) = \frac{1}{2}x^2 + x^3 - \frac{2}{3}x^5$

(h) $f(x) = 0.3x^3 + 0.12x^2 - x$

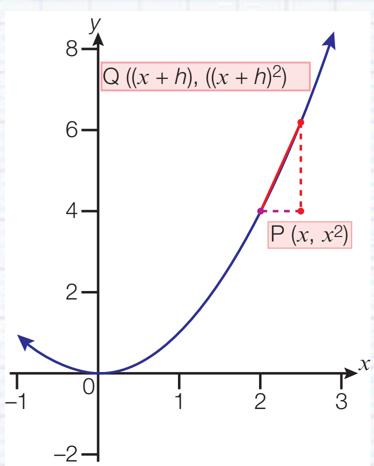
Learning links

20A Differentiation from first principles

As shown above, the gradient function of a curve can be found by taking different points on the curve (corresponding to different values of x) and calculating the gradients of shorter and shorter chords starting from each point. This method is called 'numerical differentiation', because you use actual numbers in the calculations.

You can follow a similar procedure but put letters in place of the numbers, by using algebra. This method is referred to as **differentiation from first principles.**

For example, on the curve $y = x^2$, take a general point P(x, x^2) and another point which is a distance h from P in the horizontal direction; it will be the point Q(($x + h$), $(x + h)^2$):



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The gradient of PQ is:

$$\frac{(x+h)^2 - x^2}{x+h-x}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h}$$

$$= 2x + h$$

Moving Q closer to P is the same as letting h get smaller and smaller. This is written as '$h \to 0$' in mathematical notation (pronounced as 'h tends to zero'). So, when $h \to 0$, $2x + h$ approaches $2x$. Therefore we can say that the gradient of the curve $y = x^2$ at the point (x, x^2) is $2x$, or $\frac{dy}{dx} = 2x$.

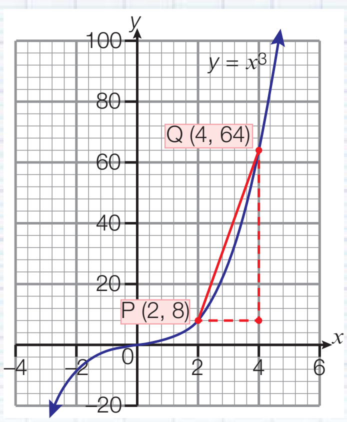
This is not covered in the syllabus, but you might see in other books the notation $\frac{dy}{dx} = \lim_{h\to 0}(2x+h) = 2x$ this just means that the derivative has been calculated using first principles, i.e. 'as h tends to zero'.

exam tip

Differentiation from first principles will not be examined, but it helps to understand the basic idea, which is very important in the development of calculus.

The number h is brought closer and closer to zero but can never equal zero, because when $h = 0$ the expression for the gradient of PQ, which is $\frac{f(x+h)-f(x)}{h}$, will have a zero in the denominator. This is an important issue, referred to as the understanding of limits, which neither Newton nor Leibniz addressed in their work.

1. The diagram shows a curve with two points P(2, 8) and Q(4, 64) marked on it.



Complete the table below by following these steps:

- Find the gradient of the chord PQ.
- Keeping P fixed at (2, 8), change the coordinates of Q progressively by reducing the x-coordinate from $x = 4$ down to $x = 2.001$.

continued . . .

- Find the gradient of each chord PQ as Q gets closer to P.

P	Q	Gradient of PQ
2	4	$\frac{64-8}{4-2} = 28$
2	3.5	$\frac{42.875-8}{3.5-2} = 23.25$
2	3	$\frac{27-8}{3-2} = 19$
2	2.5	
2	2.3	
2	2.1	
2	2.01	
2	2.001	
2	2.0001	

(a) As Q moves closer to P, the gradient of the line PQ becomes closer to the value __________.

(b) The gradient of the tangent to the curve at $x = 2$ is __________.

2. Repeat the process in question 1 for the function $y = x^2 + x$, using the initial coordinates P(3, 12) and Q(4, 20).

(a) Draw a sketch of the graph $y = x^2 + x$.

(b) Copy and complete the table below.

P	Q	Gradient of PQ
3	4	$\frac{20-12}{4-3} = 8$
3	3.5	$\frac{15.75-12}{3.5-3} = 7.5$
3	3.4	$\frac{14.96-12}{3.4-3} = 7.4$
3	3.2	
3	3.1	
3	3.01	
3	3.001	
3	3.0001	

(c) As Q moves closer to P, the gradient of the line PQ becomes closer to the value __________.

(d) The gradient of the tangent to the curve at $x = 3$ is __________.

Learning links

20B Rules of indices

In the expression x^n, n is referred to as the power, exponent, or index (plural: indices). Some properties of indices that are used in proving that the differentiation rule $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ holds for $n \in \mathbb{Q}$ are the following:

- $x^0 = 1$
- $x^n \times x^m = x^{m+n}$
- $x^n \div x^m = x^{n-m}$
- $(x^n)^m = x^{nm}$
- $\frac{1}{x^n} = x^{-n}$
- $x^{\frac{1}{n}} = \sqrt[n]{x}$

Differentiation in more detail

So far, we have seen the general rule for differentiating powers of x by looking at positive integer powers. However, the rule can be proved to hold for all power functions x^p where p can be zero, negative, a rational number, or even an irrational number.

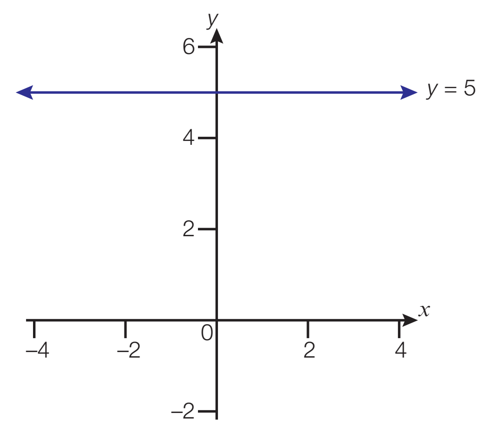
Differentiation should also confirm some geometrical facts that you already know.

Differentiating constants

To differentiate $y = 5$, note that since $x^0 = 1$, we can rewrite the function as $y = 5x^0$.

Then, using the general rule gives $\frac{dy}{dx} = 0 \times 5x^{0-1} = 0$.

Let's look at the graph of $y = 5$:



A horizontal line has a gradient of zero, so differentiation has confirmed something that you already know, that $y = 5$ is a horizontal line.

The graph of $y = c$, where c is any constant, will be a horizontal line with gradient zero. So whenever you differentiate a constant, you get zero:

If $y = c$, then $\frac{dy}{dx} = 0$.

Differentiating a straight line

To differentiate $y = 3x$, note that since $x = x^1$, we can rewrite the function as $y = 3x^1$.

Then, using the general rule gives $\frac{dy}{dx} = 1 \times 3x^{1-1} = 3x^0 = 3$.

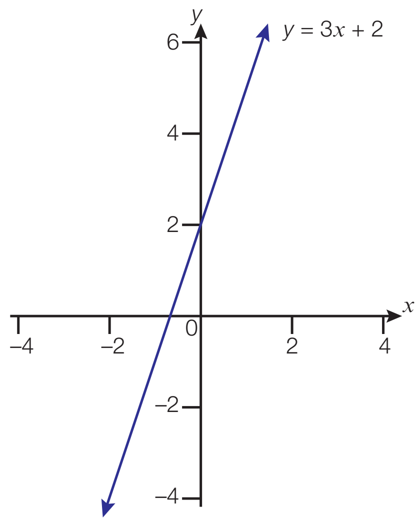
So $y = 3x$ is a line with a gradient of 3.

RR ***You learned about $y = mx + c$ in Chapter 14.***

Compare this with the equation $y = mx + c$ which you are already familiar with, where the coefficient of 'x' gives the gradient of the line.

Now let's differentiate $y = 3x + 2$. If we rewrite it as $y = 3x^1 + 2x^0$, then applying the general rule gives $\frac{dy}{dx} = 1 \times 3x^{1-1} + 0 \times 2x^{0-1} = 3 + 0 = 3$.

Again, comparing this with $y = mx + c$ confirms that $m = 3$ is the gradient of the line. It also tells us that the y-intercept, 2 in this case, has no effect on the gradient; it simply positions the line relative to the coordinate axes.



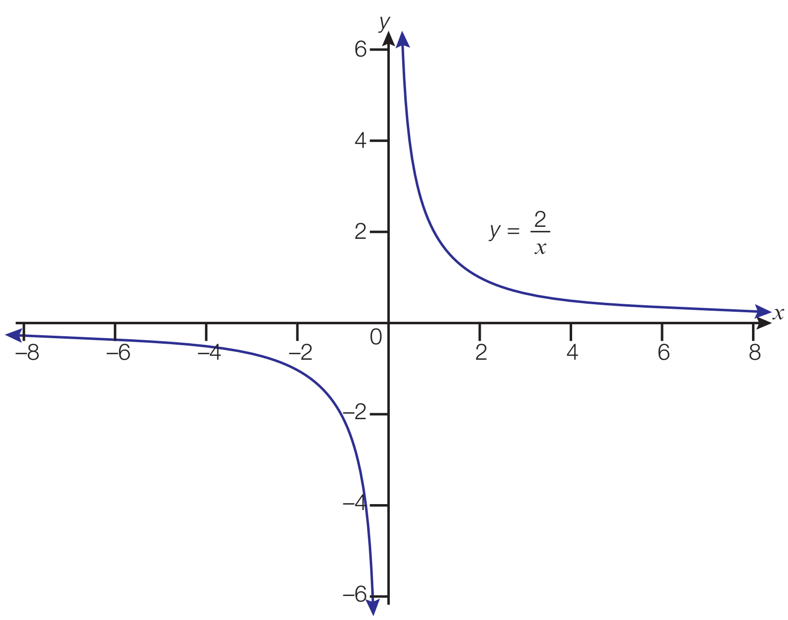
Differentiating a rational function

To differentiate the rational function $y = \frac{2}{x}$, rewrite it as a power function with a negative power: $y = \frac{2}{x} = 2x^{-1}$.

The general rule then gives $\frac{dy}{dx} = -1 \times 2x^{-1-1} = -2x^{-2} = \frac{-2}{x^2}$.

The general rule is used in exactly the same way for a negative index as it is for a positive index.

The gradient function $\frac{-2}{x^2}$ shows that the gradient of the curve will always be negative. If you look at the graph, notice that both of the curves that make up the graph are always sloping downwards.



Worked example 20.2

Q. The function $f(x)$ is given by $f(x) = x^4 - 3x^3 + 5x + 7$.

(a) Find $f'(x)$.

(b) Calculate the value of $f'(x)$ when $x = -1$.

Differentiate each term separately according to the general formula on page 580. Differentiating the constant 7 gives 0.

A. (a) $f'(x) = 4x^3 - 9x^2 + 5$

The derivative $f'(x)$ is itself a function of x, so you can evaluate it at any given x value. Substitute in $x = -1$.

(b) When $x = -1$,

$$f'(-1) = 4(-1)^3 - 9(-1)^2 + 5$$
$$= -4 - 9 + 5 = -8$$

In examinations, several different instructions may be used that all mean 'find $\frac{dy}{dx}$':

Instruction	Example function given	Example answer
Find $f'(x)$	$f(x) = 3x^3 - x + 2$	$f'(x) = 9x^2 - 1$
Differentiate with respect to x	$y = 5 - x^2$	$\frac{dy}{dx} = -2x$
Find the gradient function	$g(t) = 4t^2 + 3t$	$g'(t) = 8t + 3$
Find the derivative of the function	$h(x) = 9x - 2x^4$	$h'(x) = 9 - 8x^3$

Worked example 20.3

Q. The equation of a curve is given as $y = 2x^3 - 5x^2 + 4$

(a) Find $\frac{dy}{dx}$.

(b) Copy and complete the table below.

x	-1	0	1	2	3
y	-3	4		0	13
$\frac{dy}{dx}$		0	-4		24

(c) What is the gradient of the curve when $x = 2$?

(d) Use the table to sketch the graph of the curve.

continued . . .

Differentiate term by term using the general formula on page 580.

A. (a) $\frac{dy}{dx} = 6x^2 - 10x$

To find the missing entry in the second row, substitute that value of x into the equation for the curve.

To find the missing entries in the third row, substitute the corresponding values of x into the equation for $\frac{dy}{dx}$.

(b)

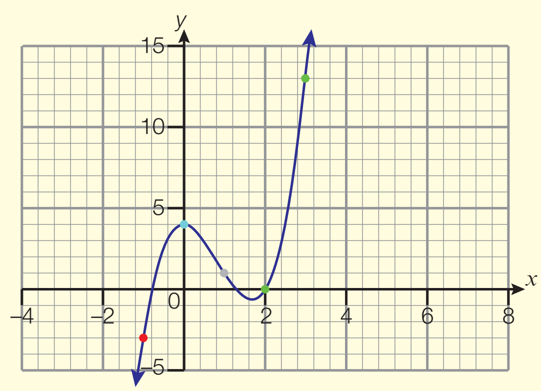
x	−1	0	1	2	3
y	−3	4	1	0	13
$\frac{dy}{dx}$	16	0	−4	4	24

The gradient of the curve when $x = 2$ is the value of $\frac{dy}{dx}$ at $x = 2$. You can read this off from the table in part (b).

(c) When $x = 2$, $\frac{dy}{dx} = 4$

Use the first two rows of the table to plot points on the curve; the third row tells you how steep the curve should be at those points.

(d)



Worked example 20.4

Q. (a) Write $f(x) = \frac{3}{x^2}$ in the form $f(x) = 3x^n$.

Use this form to differentiate $f(x)$.

(b) Write $f(x) = \frac{1}{2x^3}$ in the form $f(x) = \frac{1}{2}x^n$.

Use this form to differentiate $f(x)$.

A power of x in the denominator can be written as a negative power and then differentiated using the formula on page 580.

The power can be treated in the same way as in part (a), but in (a) the constant 3 was in the numerator, and here we have a coefficient 2 in the denominator instead.

Take extra care when working with coefficients and constants in the denominator. When differentiating, the value of the power has to multiply the fraction $\frac{1}{2}$, not the 2 in the denominator. Also be careful not to accidentally switch the numerator and denominator, which is a common mistake.

continued . . .

A. (a) $f(x) = \dfrac{3}{x^2} = 3x^{-2}$

$f'(x) = -2 \times 3x^{-2-1} = -6x^{-3}$

or $f'(x) = \dfrac{-6}{x^3}$

(b) $f(x) = \dfrac{1}{2x^3} = \dfrac{1}{2}x^{-3}$

$f'(x) = -3 \times \dfrac{1}{2}x^{-3-1}$

$= -\dfrac{3}{2}x^{-4}$

$= -\dfrac{3}{2} \times \dfrac{1}{x^4}$

$= \dfrac{-3}{2x^4}$

Exercise 20.2

1. Find $\dfrac{dy}{dx}$ for each of the following functions.

(a) $y = \dfrac{5}{7x}$ (b) $y = 2x^5 - x^{-2}$

(c) $y = x^3 - \dfrac{9}{8x^2}$ (d) $y = x - \dfrac{3}{4x^2}$

(e) $y = 3x^{-5} - 11x^2$

2. Find $f'(x)$ for each of the following functions.

(a) $f(x) = x^8 + x^{-5} + 6$ (b) $f(x) = 8x^4 - 2x^3 + x^{-4} + 13x$

(c) $f(x) = \dfrac{1}{x^3} - 10x + 2$ (d) $f(x) = \dfrac{5}{x^7} + 4x - 13$

(e) $f(x) = 9x - \dfrac{3}{5x}$

3. Differentiate the following with respect to x.

(a) $1 + x - 3x^3 + 5x^5 - 7x^7$ (b) $3 - 4x^{-2}$

(c) $10x + 9x^{-3} + \dfrac{2}{x}$ (d) $\dfrac{3}{7}x^2 + \dfrac{5}{x^2}$

(e) $\dfrac{1}{x^3} - 4x^{-5}$

4. A function is defined as $f(x) = 2x^3 - 7x^2 - 4x + 9$

(a) Find the gradient function $f'(x)$.

(b) Find the gradient of the function when:

(i) $x = 1$

(ii) $x = -2$

(iii) $x = 0$

(c) Find the value of $f'(-1)$ and explain what your answer represents.

5. The equation of a curve is defined as $y = 10 + 8x - 2x^3$

(a) Find $\frac{dy}{dx}$.

(b) Calculate the gradient of the function when:

(i) $x = 0$

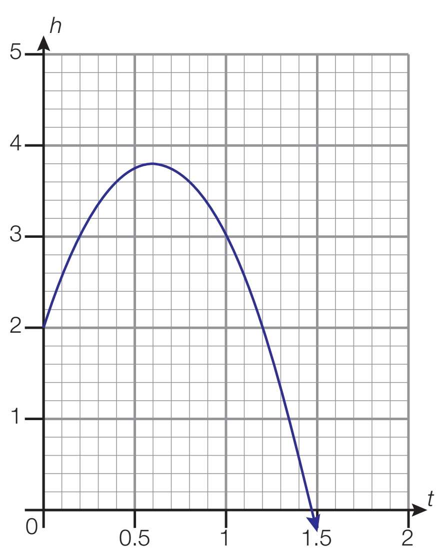
(ii) $x = -2$

(iii) $x = 3$

20.3 Rates of change

Leibniz defined the derivative by calculating how the gradient of a curve changes. He took a graphical approach that was based on the rate at which 'y' changed in relation to a small change in 'x'. Newton's development of calculus was built on his work with rates of change in physics; he used variables other than 'x' and 'y'.

As you have seen in Chapter 18, the height of a ball that has been thrown can be modelled by a quadratic function of time. Here is an example of a graph of height plotted against time:



The curve gets flatter near the top, showing that the ball slows (decelerates) as it goes higher; it stops for an instant at the maximum height, where the derivative (gradient of the curve) is zero, and then descends, gaining speed (accelerating) as it does so, which is shown by the curve getting steeper.

If the height of the ball is described by the equation $h = 2 + 6t - 5t^2$, then the rate at which the height is changing is given by $\frac{dh}{dt}$. So

$h = 2 + 6t - 5t^2$ gives the height of the ball at time t;

$\frac{dh}{dt} = 6 - 10t$ gives the rate at which the height is changing at time t.

Other variables are used in economics. For example, an equation can be used to describe the total cost (C_T) of manufacturing a certain quantity q of a product.

The **marginal cost** (C_M) is the change in total cost resulting from a small change in output.

If $C_T = 6q^2 - 7q + 10$

then $C_M = \frac{dC_T}{dq} = 12q - 7$

Whenever you want to find the rate of change of one quantity against another, you can use calculus. In any context, it is important to know what is the independent variable (x, t, q in our examples, or something else) and what is the dependent variable (y, h, C_T in our examples, or something else).

For instance, if you are studying the rate at which a plant grows, the height of the plant might be the dependent variable, and time will be the independent variable.

Worked example 20.5

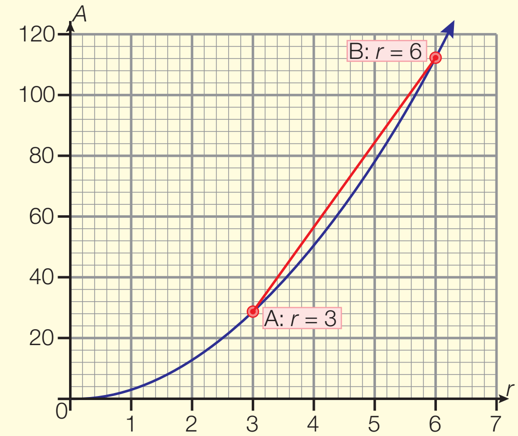
Q. Ped throws a stone into a pond and watches the circular ripples spread out from the centre. The area of a circular ripple is given by the formula $A = \pi r^2$.

(a) What is the average change in ripple area as the radius changes from 3 m to 6 m?

(b) What is the rate of change of the ripple area when the radius is exactly 4 m?

continued . . .

Sketch the graph of A against r.

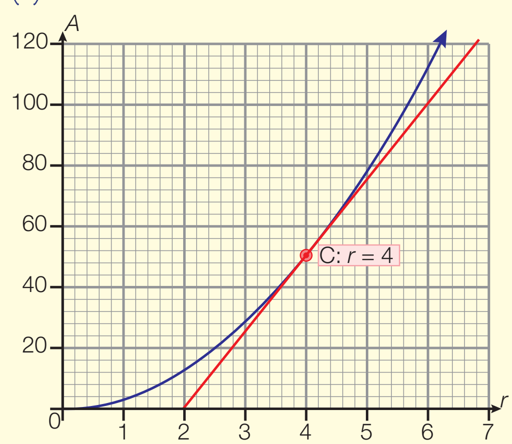


The average change in area as the radius changes is given by $\frac{\text{change in area}}{\text{change in radius}}$. On the graph, this is the gradient of the **chord** AB.

A. (a) Average change in area $= \frac{\pi \times 6^2 - \pi \times 3^2}{6-3}$

$= 9\pi$

(b)



The rate of change when r is exactly 4 is given by the gradient of the **tangent** at point C.

This is just the derivative $\frac{dA}{dr}$, evaluated at $r = 4$.

$\frac{dA}{dr} = 2\pi r$

When $r = 4$, $\frac{dA}{dr} = 2\pi \times 4 = 8\pi$

exam tip

The rate of change in part (b) of Worked example 20.5, i.e., the gradient of the tangent at a point, can also be called the '**instantaneous** rate of change'.

Exercise 20.3

1. The displacement s, in metres, of a particle t seconds after leaving point O is given by:

   $s(t) = 3t^2 - 5t + 6$

   (a) Find the average change in displacement between $t = 1$ and $t = 4$.

   (b) What is the rate of change of displacement when $t = 5$?

   (c) Calculate $s'(3)$ and explain what your answer represents.

2. A football is kicked vertically upwards. Its height h above the ground after t seconds is described by $h = 14t - 10t^2$.

   (a) Differentiate h with respect to t to find $\frac{\mathrm{d}h}{\mathrm{d}t}$.

   (b) Given that $\frac{\mathrm{d}h}{\mathrm{d}t}$ represents the velocity of the ball at a given instant, find the velocity of the ball when:

   (i) $t = 0.5$

   (ii) $t = 0.7$

3. The surface area of a circular pool of water is spreading uniformly. The area is $A = \pi r^2$, where r is the radius of the circle in metres.

   (a) Find the average rate of change of the area as the radius changes from 2 m to 4 m.

   (b) Determine the rate of change of the area when the radius is exactly 5 m.

According to Simon Singh, in his book ***Fermat's Last Theorem***, (London: Fourth Estate, 1997) 'Economics is a subject heavily influenced by calculus. Inflation is the rate of change of price, known as the derivative of price, and … the rate of change of inflation [is] known as the second derivative of price …'

The mathematician Hugo Rossi once observed the following: 'In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used a third derivative to advance his case for re-election.' (Hugo Rossi, Notices of the American Mathematical Society (Vol 43, Number 10, 1996).)

20.4 The second derivative

We have seen that the derivative, or gradient, of a curve is itself a function of the independent variable. As it is a function, it too can be differentiated. The method is exactly the same as for finding the first derivative, but the result has a different meaning and is called a 'second derivative'.

In physics, **velocity** is the rate of change of the distance moved in a certain direction, i.e. the **displacement**; in other words, velocity is the derivative of displacement with respect to time. The rate of change of velocity is called **acceleration**, so acceleration is the second derivative of displacement with respect to time.

For example, if the distance travelled by an object is given by the function:

$s = 3t^2 - 2t^3 + 1$

then the velocity is:

$$v = \frac{ds}{dt} = 6t - 6t^2$$

and the acceleration is:

$$a = \frac{d^2s}{dt^2} = 6 - 12t$$

(the expression $\frac{d^2s}{dt^2}$ is pronounced as 'D two S by D T squared').

If distance is measured in metres and time in seconds, then velocity is measured in metres per second ($m\,s^{-1}$) and acceleration is measured in metres per second per second ($m\,s^{-2}$).

Using Newton's notation, we have the displacement $s = f(t)$, the velocity $v = f'(t)$ and the acceleration $a = f''(t)$.

exam tip

Examiners will not assume that students have knowledge of the second derivative. You may use the second derivative to answer examination questions though, as long as your working makes it clear that you understand why you are using the second derivative and you have given a clear result.

Worked example 20.6

Q. A ball is thrown from the top of a cliff. Its height in metres above the cliff-top t seconds after being thrown is given by $h = 1.5 + 3t - 5t^2$. Find the equations for the velocity and acceleration of the ball, and interpret their meaning.

Differentiate the height function to get the velocity.

A. $h = 1.5 + 3t - 5t^2$

$$\text{Velocity} = \frac{dh}{dt} = 3 - 10t$$

When t is very small, the velocity is positive, which means that the ball is initially moving upwards. When t gets bigger than 0.3, the velocity becomes negative, which means that after 0.3 seconds the ball is falling downwards.

Differentiate the velocity to get acceleration.

$$\text{Acceleration} = \frac{d^2h}{dt^2} = -10$$

The acceleration has a constant magnitude of $10\,m\,s^{-2}$, and the negative sign tells us that its direction is downward. This is the acceleration due to gravity.

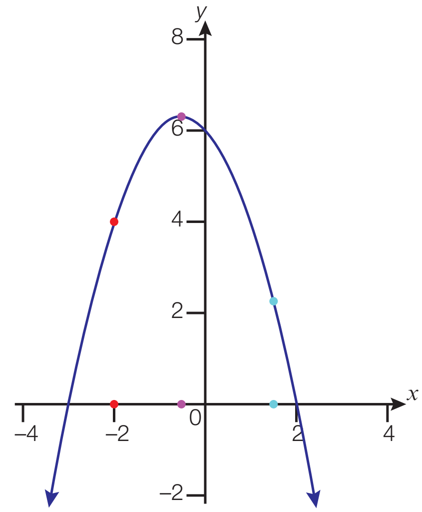
20.5 Gradient of a curve at a given point

Differentiation can give you the gradient function for any curve.

By using the formula given in section 20.2, you can find the gradient function of any curve with an equation of the form $f(x) = ax^n + bx^m + \ldots$.

It is possible to differentiate far more complex curves, but those methods are not in the syllabus for this course.

The following curve is the graph of $f(x) = 6 - x - x^2$.



Differentiate to find the equation of the gradient function:

$$f'(x) = -1 - 2x$$

Using this equation you can calculate the gradient at any particular point on the curve. For instance:

When $x = -2$, $f'(x) = -1 - 2 \times (-2) = -1 + 4 = 3$; the gradient is positive.

When $x = -\frac{1}{2}$, $f'(x) = -1 - 2 \times \left(-\frac{1}{2}\right) = -1 + 1 = 0$; the gradient is zero.

When $x = 1.5$, $f'(x) = -1 - 2 \times 1.5 = -1 - 3 = -4$; the gradient is negative.

Compare these results with the graph. Notice that as x increases, the gradient changes from positive values, through zero, to negative values.

Finding x- and y-coordinates from the gradient

It is also possible to work backwards: if you are given a specific value of the gradient, you can use the gradient function to find the x-coordinate(s) of the point(s) on the curve with that gradient. Once you know the x-coordinate, you can use the equation of the curve to calculate the corresponding value of y.

Consider the function $f(x) = 6 - x - x^2$ again. What are the coordinates of the point where the curve has a gradient of 4?

The gradient function is $f'(x) = -1 - 2x$, so you are looking for the x value that makes this equal to 4:

$$-1 - 2x = 4$$

so $\quad x = -2.5$

For $x = -2.5$, use the equation $f(x) = 6 - x - x^2$ to find the corresponding y-coordinate:

$$f(-2.5) = 6 - (-2.5) - (-2.5)^2 = 6 + 2.5 - 6.25 = 2.25$$

The curve $f(x) = 6 - x - x^2$ has a gradient of 4 at the point $(-2.5, 2.25)$.

Worked example 20.7

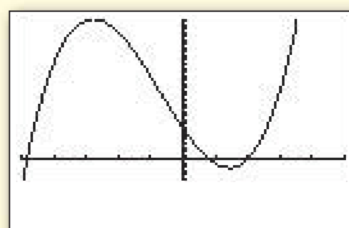
Q. Consider the function $f(x) = x^3 + 2x^2 - 12x + 8$. Find:

(a) $f'(x)$

(b) $f'(2)$ (the value of $f'(x)$ when $x = 2$)

(c) the coordinates of the point where the gradient is 27.

Sketch the curve using either your GDC or a computer, it will help you to understand the rest of the question. (See '*22.2G Graphs*' on page 645 of the GDC chapter if you need a reminder.)



Differentiate term by term using the formula on page 580.

A. (a) $f'(x) = 3x^2 + 4x - 12$

Substitute $x = 2$ into the gradient function.

(b) $f'(2) = 3 \times 2^2 + 4 \times 2 - 12 = 12 + 8 - 12 = 8$

You want to find the value of x for which $f'(x) = 27$.

(c) $3x^2 + 4x - 12 = 27$

Rearrange into the general form of a quadratic equation ($ax^2 + bx + c = 0$).

$3x^2 + 4x - 39 = 0$

Use your GDC (or other method) to solve the equation. (See Chapter 2 for a reminder of methods.)

The question asks for the **coordinates** of the points, so use $f(x) = x^3 + 2x^2 - 12x + 8$ to find the y-coordinate corresponding to each x value.

continued . . .

From GDC: $x = 3$ or $x = -4.33$

When $x = 3$, $y = 3^3 + 2 \times 3^2 - 12 \times 3 + 8 = 17$

When $x = -4.33$, $y = (-4.33)^3 + 2 \times (-4.33)^2 - 12 \times (-4.33) + 8 = 16.2$

The points are (3, 17) and (−4.33, 16.2).

Worked example 20.8

Q. As part of their Business Studies course, Myra and Salim have set up a company to manufacture scarves. They calculate that the total cost of production (C_T), in US dollars, is given by the function:

$$C_T = 2q^2 - 6q + 5$$

where q is the number of scarves produced.

(a) Find the value of the marginal cost, $\frac{dC_T}{dq}$, when $q = 25$. (The marginal cost can be thought of as the additional cost of making one extra scarf above the current quantity.)

(b) If the marginal cost is \$42, how many scarves are they producing?

Differentiate C_T to get the marginal cost function.

A. (a) $\frac{dC_T}{dq} = 4q - 6$

When $q = 25$, the marginal cost is

$4 \times 25 - 6 = \$94$

Now we want to find the value of q for which $\frac{dC_T}{dq} = 42$.

(b) $4q - 6 = 42$

$4q = 48$

$q = 12$

They are making 12 scarves.

Using your GDC for differential calculus

Your GDC cannot differentiate a function for you, but there are many ways in which your GDC can help you with questions involving gradients. Here are some ideas.

Your GDC can:

- draw the curve, so that you can see how the gradient changes
- get the **value** of $\frac{dy}{dx}$ for a given value of x
- find the y-coordinate corresponding to any given x-coordinate
- solve any equations that arise in the problem.

Worked example 20.9

Q. (a) Differentiate the following function with respect to x:

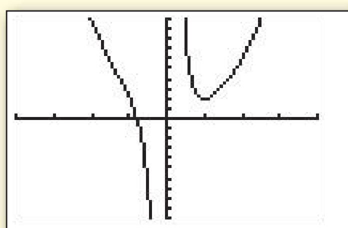
$$f(x) = 2x^2 - x + \frac{1}{x}$$

(b) Calculate $f'(3)$.

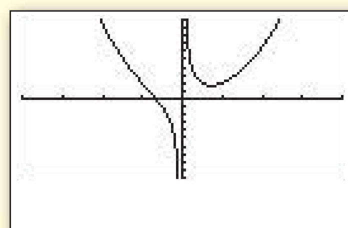
(c) Find the value of x at the point where the gradient of the curve is 2.

Draw the curve on your GDC to get an idea of what you're dealing with. (See '*22.2G Graphs*' on page 645 of the GDC chapter if you need a reminder of how.)

TEXAS



CASIO



Rewrite $f(x)$ as a sum of powers of x.

A. (a) $f(x) = 2x^2 - x + x^{-1}$

Differentiate term by term using the formula on page 580.

Your calculator can't help you with this!

$$f'(x) = 4x - 1 - x^{-2}$$

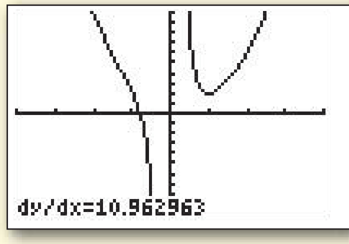
$$= 4x - 1 - \frac{1}{x^2}$$

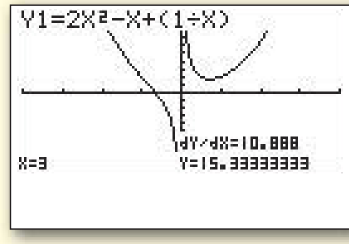
Calculating $f'(3)$ is the same as finding $\frac{dy}{dx}$ when $x = 3$. You can substitute $x = 3$ into the derivative function found in part (a) or use your GDC. See '*20.1 (a) Finding the numerical value of* $\frac{dy}{dx}$ *using a graph*' on page 686 of the GDC chapter if you need to .

continued . . .

(b)

TEXAS | **CASIO**





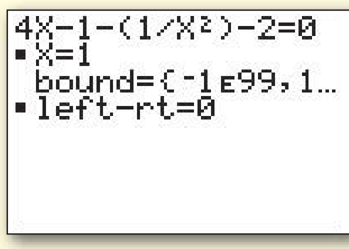
When $x = 3$, $\frac{dy}{dx} = 10.9$

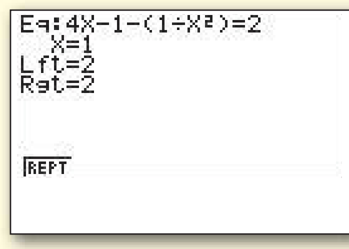
To find the value of x where the gradient is 2, you need to solve the equation $f'(x) = 2$.

(c) $4x - 1 - \frac{1}{x^2} = 2$

You can use the equation solver on your GDC. *See '19.2 (b) Solving unfamiliar equations using the equation solver'* on page 685 of the GDC chapter if you need to.

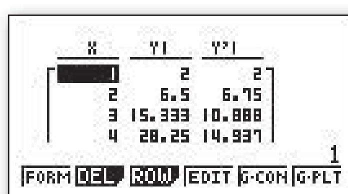
TEXAS | **CASIO**





The gradient of the curve is 2 at the point where $x = 1$.

If you are asked for the coordinates of the point where the gradient has a particular value, you can find this information using the table function on your GDC (see '*20.1 (b) Finding the numerical value of the derivative* ($\frac{dy}{dx}$) *using the table*' on page 687 of the GDC chapter of you need a reminder of how to do this). For part (c) of Worked example 20.9, the table on a CASIO calculator would look like this:



From the table you can see that when the gradient (third column) is 2, the x-coordinate is 1 (first column) and the y-coordinate is 2 (second column). So (1, 2) is the point on the curve where the gradient is 2.

Note that on a TEXAS GDC, the table will not show values of the gradient in the third column. In this case, you would need to find the value of x for a given gradient, and then look up the corresponding y value in the table.

Exercise 20.4

1. Find the gradient of the following curves at the points with the specified x-coordinates:

   (a) $f(x) = 3x^4$, when $x = 5$

   (b) $g(x) = -12x^3$, when $x = 1$

   (c) $h(x) = x^2 - 13x$, when $x = 0$

   (d) $y = 8x - \frac{5}{6}x^2$, when $x = -6$

   (e) $y = x^2 - 10x + 7$, when $x = 3$

   (f) $y = 5 + 6x - 4x^3$, when $x = -2$

   (g) $f(x) = 7 - 8x^2 - 2x^3$, when $x = -1$

   (h) $f(x) = 11 - 2x^2 + 3x^4$, when $x = \frac{1}{2}$

   (i) $y = 9 - 8x^2 + \frac{2}{3}x^3$, when $x = 4$

   (j) $y = \frac{5}{x^2}$, when $x = 1$

   (k) $f(x) = 3x + \frac{12}{x^4}$, when $x = -3$

   (l) $y = \frac{5}{x^3} + \frac{1}{4x}$, when $x = 2$

2. In the following questions you are given the equation for the total cost of production, C, for a quantity of items, q.

   In each case:

   (i) Work out an equation for the marginal cost by differentiating the total cost with respect to q; that is, find $\frac{dC}{dq}$.

   (ii) Determine the marginal cost for the stated value of q.

   (a) $C(q) = 8q^2 - 9$, when $q = 10$

   (b) $C(q) = 300 + 5q^2$, when $q = 120$

   (c) $C(q) = 70 + 5q + 3q^2$, when $q = 80$

   (d) $C(q) = 3q^2 - 10q + 64$, when $q = 14$

   (e) $C(q) = 2q^3 - 9q^2 + 45q + 7$, when $q = 200$

3. The equation defining a function is $y = x^2 - 4x - 12$.

   (a) Differentiate the equation to find the gradient function, $\frac{dy}{dx}$.

   (b) Find the gradient of the curve at the point where the x-coordinate is 3.

   (c) Find the value of x where the gradient of the curve is 8.

4. A function f is defined as $f(x) = x^3 - 4x^2 + 8$.

   (a) Work out $f'(x)$ in terms of x.

   (b) Find $f'(-2)$.

   (c) Find the coordinates of the points on the curve where the gradient is -4.

5. The equation of a curve is $y = 2x + \frac{1}{x}$.

   (a) Find $\frac{dy}{dx}$.

   (b) Find the gradient of the curve at the point where the x-coordinate is -1.

   (c) Find the coordinates of the points on the curve where the gradient is -7.

6. The total cost (in dollars) of manufacturing q items of a certain product is given by:

   $$C_T = 800 - 5q + \frac{2}{3}q^2$$

   (a) Find the marginal cost, $\frac{dC_T}{dq}$, when 90 items are being produced.

   (b) Find the number of items being produced when the marginal cost is \$55.

7. Given that $s = 28t - 10t^2$, where s is the displacement in metres and t is the time in seconds, find an expression for:

   (a) the velocity of the particle $(v = \frac{ds}{dt})$

   (b) the acceleration of the particle $(a = \frac{d^2s}{dt^2})$.

20.6 Equation of the tangent at a given point

The tangent to a curve at a particular point has been defined as a straight line that touches the curve at that point.

It is possible to find the equation of any line provided that you know its **gradient** and the **coordinates of one point** that it passes through. So, you can find the equation of the tangent to a curve at a given point using methods that are already familiar.

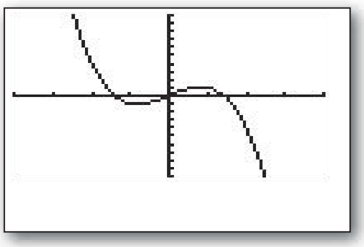
If you are given the x-coordinate of the point, you can use the equation of the curve to find the y-coordinate. The gradient of the tangent is just the gradient of the curve at that point; so to find the gradient, differentiate the equation of the curve and calculate the value of the derivative at the given x value.

For example, to find the equation of the tangent to the curve $y = 2x - x^3$ at the point where $x = 2$:

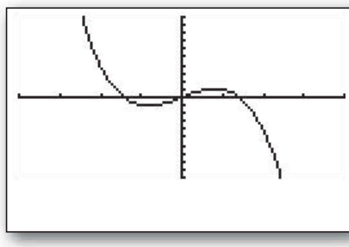
- Calculate the y-coordinate. If $x = 2$, then $y = 2 \times 2 - 2^3 = 4 - 8 = -4$
- Differentiate to get the gradient function. $\frac{dy}{dx} = 2 - 3x^2$
- Find the gradient when $x = 2$. When $x = 2$, $\frac{dy}{dx} = 2 - 3 \times 2^2 =$ $2 - 12 = -10$
- Taking the general equation of a line, $y = mx + c$, put in the x- and y-coordinates and the value of the gradient (m) and solve for c:
  $-4 = -10 \times 2 + c$
  $c = 16$
- Hence the equation of the tangent is $y = -10x + 16$.

Alternatively, you can use your GDC:

TEXAS



CASIO

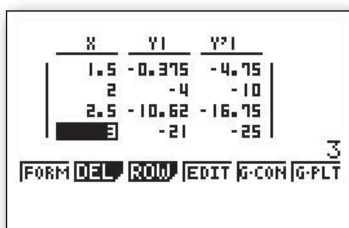


Draw the curve (see '*22.2G Graphs*' on page 645 of the GDC chapter if you need to).

TEXAS

There is not a column for y'_1 in the TEXAS table of coordinates, so substitute $x = 2$ into the gradient function to find y'_1.

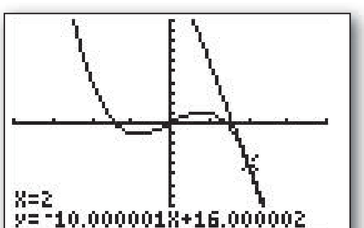
CASIO



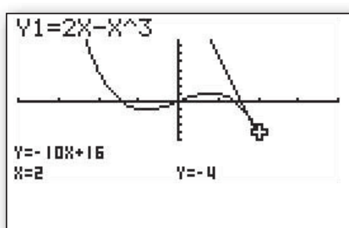
Look at the table of coordinates for the graph (see '*20.1 (b) Finding the numerical value of the derivative using a table.*' on page 686 of the GDC chapter if you need to), and find the value of y corresponding to $x = 2$. If you are using a CASIO calculator you will also be able to see that the gradient at $x = 2$ is -10.

When $x = 2$, $y = -4$

TEXAS



CASIO



Draw the tangent to the curve at the point (2, –4). See '*20.2 Finding the equation of tangents at a point*' on page 687 of the GDC chapter if you need to.

The equation of the tangent is $y = -10x + 16$.

Read off the equation of the tangent from the screen. Note that you should round the figures on the screen to whole numbers.

Worked example 20.10

We will solve this problem first using algebra and then using the GDC in two ways.

Q. Find the equation of the tangent to the curve $y = \frac{3}{x^2} - 2x$ at the point where $x = -2$.

A. Method 1:

Using algebra: rewrite the function as a sum of powers of x, and differentiate term by term using the formula on page 580.

$$y = \frac{3}{x^2} - 2x$$

$$y = 3x^{-2} - 2x$$

$$\frac{dy}{dx} = -6x^{-3} - 2$$

$$= \frac{-6}{x^3} - 2$$

Find the y-coordinate and the gradient at $x = -2$.

At $x = -2$,

$$y = \frac{3}{(-2)^2} - 2\times(-2) = 4\frac{3}{4} = \frac{19}{4}$$

$$\frac{dy}{dx} = \frac{-6}{(-2)^3} - 2 = \frac{6}{8} - 2 = -\frac{5}{4}$$

Put these values into $y = mx + c$ and solve for c.

$$y = mx + c$$

$$\frac{19}{4} = -\frac{5}{4}\times(-2) + c$$

$$\frac{19}{4} = \frac{10}{4} + c$$

$$c = \frac{9}{4}$$

Write down the equation of the tangent.

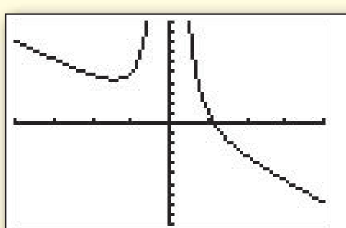
Equation of the tangent is

$$y = -\frac{5}{4}x + \frac{9}{4}$$

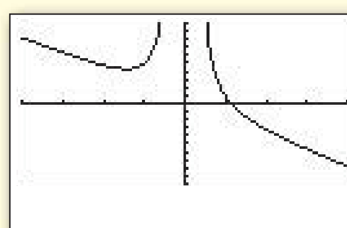
Method 2(a):

Using your GDC: draw the curve (see '*22.2G Graphs*' on page 645 of the GDC chapter if you need to).

TEXAS



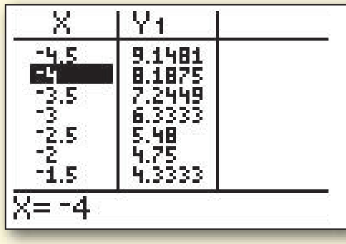
CASIO



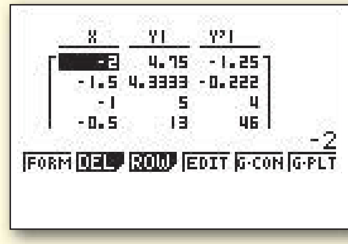
continued . . .

Look at the table of coordinates to find the y-coordinate corresponding to $x = -2$. (See '*14.1*' on page 678 of the GDC chapter if you need to.)

TEXAS



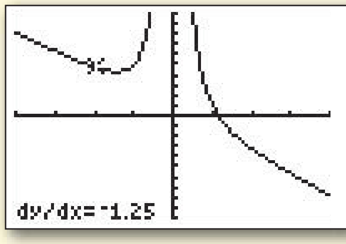
CASIO



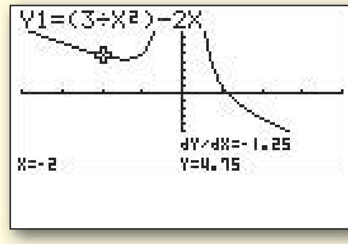
When $x = -2$, $y = 4.75$

From the graph, find the value of $\frac{dy}{dx}$ when $x = -2$. (See '*20.1 (a) Finding the numerical value of the derivative using a graph*' on page 686 of the GDC chapter if you need to.)

TEXAS



CASIO



$\frac{dy}{dx} = -1.25$

Substitute the values into $y = mx + c$ and solve for c.

$y = mx + c$

$4.75 = -1.25 \times (-2) + c$

$c = 4.75 - 2.5 = 2.25$

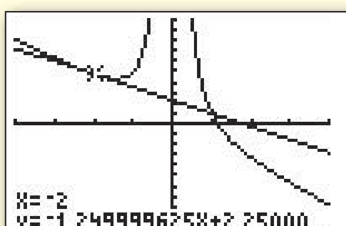
Equation of the tangent is

$y = -1.25x + 2.25$

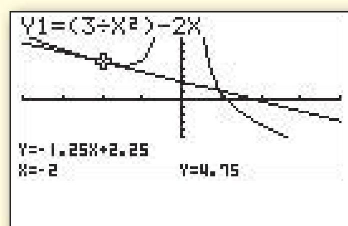
<u>Method 2(b):</u>

Using your GDC: on the graph, draw the tangent to the curve at $x = -2$. (See '*20.2 Finding the equation of a tangent at a point*' on page 687 of the GDC chapter if you need to.)

TEXAS



CASIO



Read off the equation of the tangent from the screen.

$y = -1.25x + 2.25$

Exercise 20.5

1. For each of the following curves, use your GDC to find the equation of the tangent at the point with the given x-coordinate.

(a) $y = 3x^2 - 10$, at $x = 4$

(b) $y = 4 - x^2$, at $x = -3$

(c) $f(x) = 9 - x^4$, at $x = -1$

(d) $f(x) = x - x^3$, at $x = \frac{1}{2}$

(e) $y = 2x^3 + 7x^2 - 3x + 4$, at $x = 1$

(f) $y = 2 + \frac{1}{x}$, at $x = -2$

(g) $g(x) = 8 - \frac{3}{x^2}$, at $x = 2$

(h) $y = \frac{7}{x^3} - \frac{1}{x^2}$, at $x = 1.5$

2. For each of the following curves, find the equation of the tangent to the curve at the given point.

(a) $y = x^2 - x - 12$ at $(-3, 0)$

(b) $y = 2x^3 + 3x^2 - 23x - 12$ at $(2, -30)$

(c) $y = 6x^3 - 19x^2 + 19x - 6$ at $(-\frac{1}{2}, -21)$

(d) $f(x) = 11 - 2x^2$ at $(3, -7)$

(e) $f(x) = \frac{3}{x^2} - x$ at $(\frac{1}{2}, \frac{23}{2})$

(f) $y = 1 - 2x - \frac{2}{x}$ at $(-1, 5)$

3. A function is defined as $y = 2x^3 - x^2 + 4x + 1$.

(a) Find $\frac{dy}{dx}$ in terms of x.

The point P lies on the curve. The x-coordinate at P is 2.

(b) Find the gradient of the curve at P.

(c) State the y-coordinate of P.

(d) Write down the equation of the tangent to the curve at P.

4. A function is defined as $y = 9 - \frac{x^2}{16}$.

(a) Find $\frac{dy}{dx}$ in terms of x.

The point P lies on the curve. The x-coordinate at point P is -1.

(b) Find the gradient of the curve at P.

(c) State the y-coordinate of P.

(d) Write down the equation of the tangent to the curve at P in the form $ax + by + d = 0$ where a, b and d are integers.

20.7 Equation of the normal at a given point

A straight line that makes a right angle with the tangent to a curve at a particular point is called the **normal** to the curve at that point. As the normal is perpendicular to the tangent, the rule for two perpendicular gradients can be used:

If two lines are perpendicular, their gradients multiply to give -1.

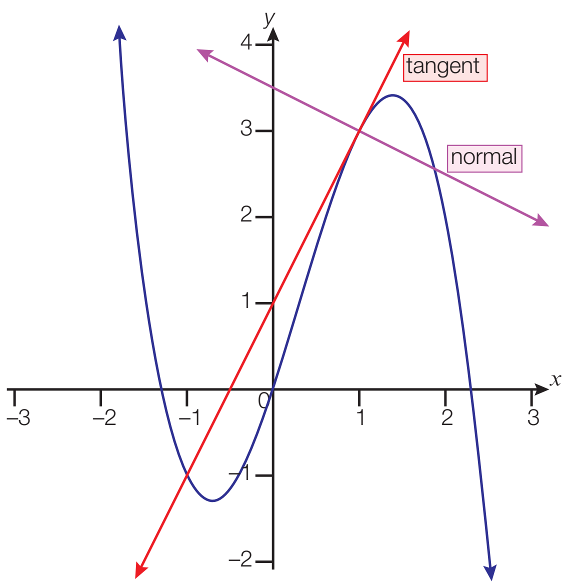
$m_1 \times m_2 = -1$

RR *This relationship was covered Chapter 14.*

So if the tangent at a point on the curve has gradient 4, the gradient of the normal will be $-\frac{1}{4}$, because $4 \times \left(-\frac{1}{4}\right) = -1$.

A quick way of finding the gradient of the normal is to **turn the gradient of the tangent upside down, and change the sign**.

For the example above, the gradient of the tangent, which is 4, can be written as the fraction $\frac{4}{1}$. Turning it upside down gives $\frac{1}{4}$, and then switching the sign gives $-\frac{1}{4}$.



The curve illustrated in the diagram above is the function $f(x) = 3x + x^2 - x^3$. The tangent and normal have been drawn at the point (1, 3).

To find the equation of the normal, first differentiate the equation of the curve to find the gradient function:

$$f'(x) = 3 + 2x - 3x^2$$

Replace x by 1 to get:

$$f'(1) = 3 + 2 \times 1 - 3 \times 1^2 = 3 + 2 - 3 = 2$$

This means that the gradient of the tangent at (1, 3) is 2.

So the gradient of the normal is $-\frac{1}{2}$.

Now we can substitute the coordinates (1, 3) and the gradient $-\frac{1}{2}$ into the equation $y = mx + c$ and solve for c:

$$3 = -\frac{1}{2} \times 1 + c$$

$$c = 3 + \frac{1}{2} = 3\frac{1}{2}$$

Therefore the equation of the normal is $y = -\frac{1}{2}x + 3\frac{1}{2}$.

You can also use your GDC to help you find the equation of the normal. Some calculators have this facility built into them (most CASIO models), in which case you would just tell it to draw the normal to the curve at a specified point, and it will display the equation of the normal.

Other models of calculator (most TEXAS models) may not be able to give you the equation of the normal directly. In this case, you could still use your GDC to find the coordinates of the point and the gradient of the tangent; then follow the procedure above to calculate the gradient of the normal and the constant c in $y = mx + c$.

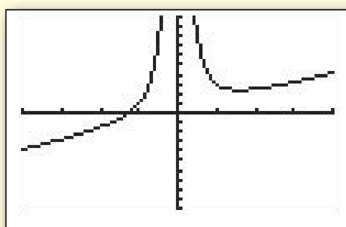
Worked example 20.11

Q. If $f(x) = \dfrac{2}{x^2} + x$, find the equation of the normal at $x = -1$.

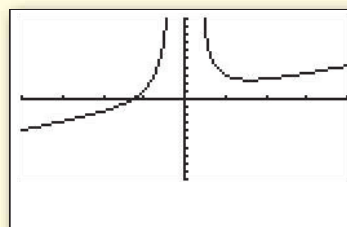
A.

Draw the curve to get an idea of its shape.

TEXAS



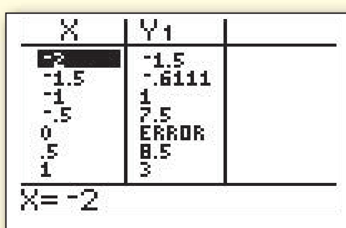
CASIO



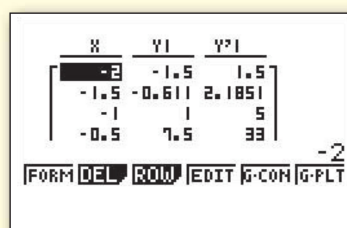
Look at the table of coordinates to find the value of the y-coordinate.

(See '*14.1 Accessing the table of coordinates from a plotted graph*' on page 678 of the GDC chapter if you need to.)

TEXAS



CASIO

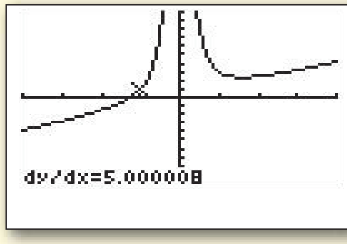


When $x = -1$, $y = 1$

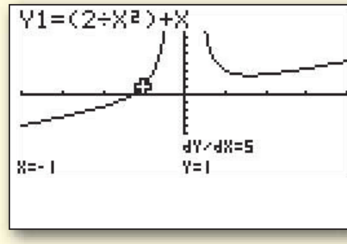
continued . . .

Find the value of $\frac{dy}{dx}$ when $x = -1$. This is the gradient of the tangent. (See '*20.2 Finding the equation of the tangent at a point*' on page 687 of the GDC chapter if you need to.)

TEXAS



CASIO



$\frac{dy}{dx} = 5$

Now calculate the gradient of the normal. $\frac{dy}{dx} = 5$, and $5 = \frac{5}{1}$; turn it upside down to get $\frac{1}{5}$, and then change the sign from positive to negative.

So the gradient of the normal is $-\frac{1}{5}$.

Substitute the coordinates and the gradient into the equation for a straight line.

$y = mx + c$

$1 = -\frac{1}{5} \times (-1) + c$

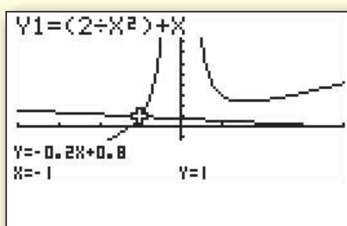
$c = 1 - \frac{1}{5} = \frac{4}{5}$

Finally, write down the equation of the normal.

The equation of the normal is $y = -\frac{1}{5}x + \frac{4}{5}$.

If your GDC is able to give you the equation of the normal directly, then you can read it off directly from the screen. (See '*20.2 Finding the equation of the tangent at a point*' on page 687 of the GDC chapter if you need to.)

CASIO



(TEXAS calculators do not have a direct function for this.)

The equation of the normal is
$y = -0.2x + 0.8$

Exercise 20.6

GDC

A normal line drawn by a GDC is likely to look distorted, and may not appear to be at right angles to the curve.

1. For each of the following curves, use your GDC to find the equation of the normal to the point with the given x-coordinate.

(a) $y = 3x^2 - 10$, at $x = 4$

(b) $y = 4 - x^2$, at $x = -3$

(c) $f(x) = 9 - x^4$, at $x = -1$

(d) $f(x) = x - x^3$, at $x = \frac{1}{2}$

(e) $y = 2x^3 + 7x^2 - 3x + 4$, at $x = 1$ (f) $y = 2 + \frac{1}{x}$, at $x = -2$

(g) $g(x) = 8 - \frac{3}{x^2}$, at $x = 2$ (h) $y = \frac{7}{x^3} - \frac{1}{x^2}$, at $x = 1.5$

2. Find the equation of the normal to the curve at the given point.

(a) $y = x^2 - x - 12$, at $(-3, 0)$

(b) $y = 2x^3 + 3x^2 - 23x - 12$, at $(2, -30)$

(c) $y = 6x^3 - 19x^2 + 19x - 6$, at $(-\frac{1}{2}, -21)$

(d) $f(x) = 11 - 2x^2$, at $(3, -7)$

(e) $f(x) = \frac{3}{x^2} - x$, at $(\frac{1}{2}, \frac{23}{2})$

(f) $y = 1 - 2x - \frac{2}{x}$, at $(-1, 5)$

3. Find the equation of the normal to the curve with equation $y = 1 - 3x^2 - x^3$ at the point where $x = -1$.

4. A curve is the graph of the function $f(x) = x^3 + 5x^2 - 2$. A point N lies on the curve. The x-coordinate of N is -4. Find the equation of the normal to the curve at the point N.

Summary

You should know:

- the two concepts of a derivative
  - as a rate of change
  - as the gradient of a graph
- how to obtain a tangent to a curve at a particular point on the curve, and how to use the gradient of the tangent to define the gradient of a curve
- the general differentiation formula $f(x) = ax^n \Rightarrow f'(x) = nax^{n-1}$
- how to calculate the derivative of a function of the form $f(x) = ax^n + bx^m + \ldots$ where all exponents are integers (positive or negative)
- how to find the gradient of a curve at a given value of x
- how to find the value(s) of x on a curve that has a given value of $f'(x)$
- how to calculate the equation of the tangent to a curve at a given point
- that the normal is the line perpendicular to the tangent at a given point
- how to calculate the equation of the normal at a given point.

Mixed examination practice

Exam-style questions

1. The equation of a curve is defined as $y = x^4 - 7x^3 - 9x + 6$. Find $\frac{dy}{dx}$.

2. The equation of a function is $y = x^2 - 8x + 7$.

(a) Differentiate the equation to find the gradient function $\frac{dy}{dx}$.

(b) Find the gradient of the curve at the point where the x-coordinate is 3.

(c) Find the value of x at which the gradient of the curve is 6.

3. The equation for the total cost of production, C_T, for a quantity of items, q, is given by:

$$C(q) = 120q - q^2 - 0.005q^3$$

(a) Find an equation for the marginal cost by differentiating the total cost with respect to q; that is, find $\frac{dC_T}{dq}$.

(b) Determine the marginal cost $\frac{dC_T}{dq}$ when $q = 40$.

4. The displacement, s, of a particle t seconds after leaving point O is described by:

$$s(t) = 12 + t - t^2$$

(a) Find the average change in displacement between $t = 1$ and $t = 2$.

(b) What is the rate of change of displacement when $t = 3$?

(c) Calculate $s'(3.5)$ and explain what your answer represents.

5. A particle moves such that its displacement, s metres, at time t seconds is given by:

$$s = 2t^3 - 4t^2 + 4t - 7$$

(a) Find an expression for:

(i) the velocity of the particle, v

(ii) the acceleration of the particle, a

(b) Calculate the velocity of the particle when:

(i) $t = 2$ (ii) $t = 4$.

(c) Determine the acceleration of the particle when:

(i) $t = 1$ (ii) $t = 4$.

(d) Find the time t when the acceleration is zero.

6. A curve has equation $f(x) = 5x^2 - 4x - \frac{3}{x}$.

(a) State the value of $f(1)$. What does $f(1)$ represent?

(b) Find the value of $f'(1)$. What does your answer represent?

The point Q lies on the curve. At Q, $x = 1$.

(c) Write down the equation of the tangent to the curve at Q.

7. A curve has equation $y = x^3 - 4x^2 - x + 5$.

(a) Find the gradient of the tangent to the curve at $x = 1$.

(b) State the gradient of the normal to the curve at $x = 1$.

The point P lies on the curve and has x-coordinate equal to 1.

(c) Find the y-coordinate of P.

(d) Write down the equation of the tangent to the curve at P.

(e) Work out the equation of the normal to the curve at P.

8. A curve has equation $y = x^2(3x - 5)$.

(a) Expand the expression $x^2(3x - 5)$.

(b) Use your answer from part (a) to find the gradient function of the curve in terms of x.

(c) Find the gradient of:

(i) the tangent (ii) the normal

to the curve at the point Q whose x-coordinate is -2.

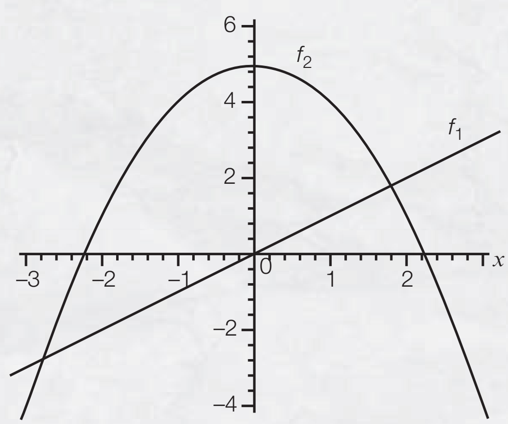
(d) Write down the equation of:

(i) the tangent (ii) the normal

to the curve at point Q.

Past paper questions

1. The figure below shows the graphs of functions $f_1(x) = x$ and $f_2(x) = 5 - x^2$.



(a) (i) Differentiate $f_1(x)$ with respect to x.

(ii) Differentiate $f_2(x)$ with respect to x.

(b) Calculate the value of x for which the gradient of the two graphs is the same.

(c) Draw the tangent to the **curved** graph for this value of x on the figure, showing clearly the property in part (b). *[Total 6 marks]*

[May 2007, Paper 1, Question 11] (© *IB Organization 2007*)

2. Consider the function $f(x) = 2x^3 - 5x^2 + 3x + 1$.

(a) Find $f'(x)$. *[3 marks]*

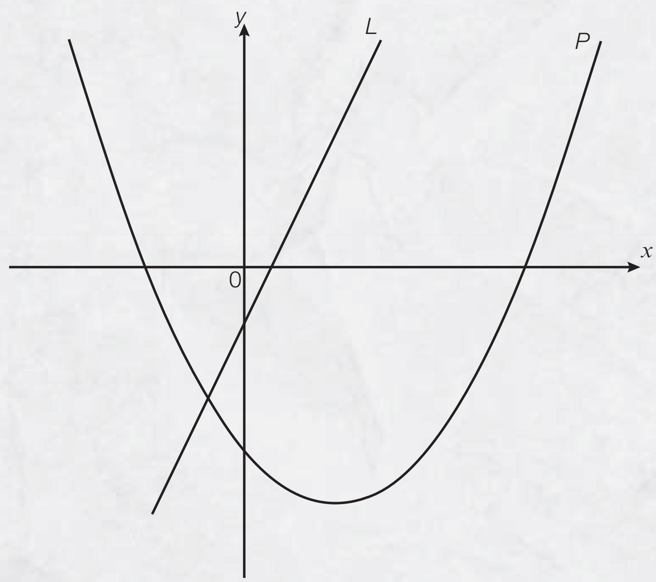
(b) Write down the value of $f'(2)$. *[1 mark]*

(c) Find the equation of the tangent to the curve $y = f(x)$ at the point (2, 3). *[2 marks]*

[Total 6 marks]

[May 2008, Paper 1, TZ1, Question 3] (© *IB Organization 2008*)

3. The diagram below shows the graph of a line L passing through (1, 1) and (2, 3) and the graph P of the function $f(x) = x^2 - 3x - 4$.



(a) Find the gradient of the line L. *[2 marks]*

(b) Differentiate $f(x)$. *[2 marks]*

(c) Find the coordinates of the point where the tangent to P is parallel to the line L. *[3 marks]*

(d) Find the coordinates of the point where the tangent to P is perpendicular to the line L. *[4 marks]*

(e) Find:

(i) the gradient of the tangent to P at the point with coordinates $(2, -6)$;

(ii) the equation of the tangent to P at this point. *[3 marks]*

(f) State the equation of the axis of symmetry of P. *[1 mark]*

(g) Find the coordinates of the vertex of P and state the gradient of the curve at this point. *[3 marks]*

[Total 18 marks]

[Nov 2007, Paper 2, TZ0, Question 5] (© *IB Organization 2007*)

Chapter 21 Stationary points and optimisation

Differential calculus provides mathematicians, scientists, economists and other technical professionals with a powerful technique for solving practical problems.

If an equation can be found that models a situation and connects some of the variables that are being studied, then the rates at which those variables change relative to each other can be analysed.

For instance:

- In medicine: What is the most efficient way of administering a drug? When is the concentration of that drug in the patient's bloodstream at its highest or lowest?
- In engineering: Where are the greatest stresses on a beam? What is the maximum load that a bridge can bear?
- In commerce: How can a company maximise its use of resources? What is the minimum quantity of material that could be ordered and still get the job done?

In Ian Stewart's book *Seventeen Equations that Changed the World*, he says:

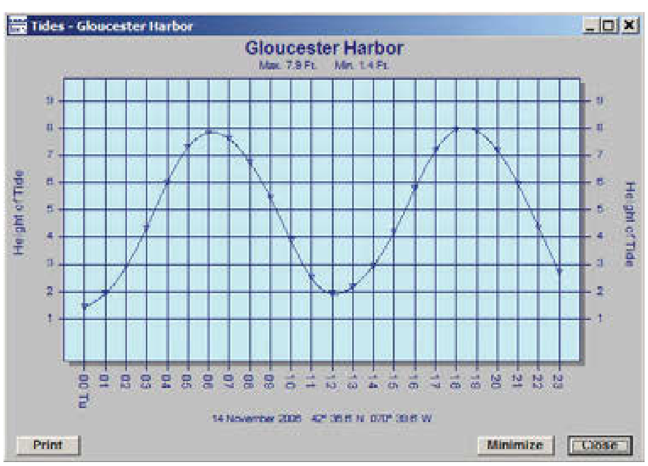
'… calculus is simply an indispensible tool in the engineer's and scientist's tool kit. More than any other mathematical technique, it has created the modern world.'

Source: Ian Stewart, *Seventeen Equations that Changed the World* (Profile Books, 2012).

In this chapter you will learn:

- about increasing and decreasing functions
- the graphical interpretation of $f'(x) < 0$, $f'(x) = 0$ and $f'(x) > 0$
- how to find values of x where the gradient of a curve is zero
- why the solution of the equation $f'(x) = 0$ is important
- what stationary points are and how to find them
- how to determine whether a stationary point is a local maximum or minimum point
- how to use calculus to solve optimisation problems.

21.1 Increasing and decreasing functions



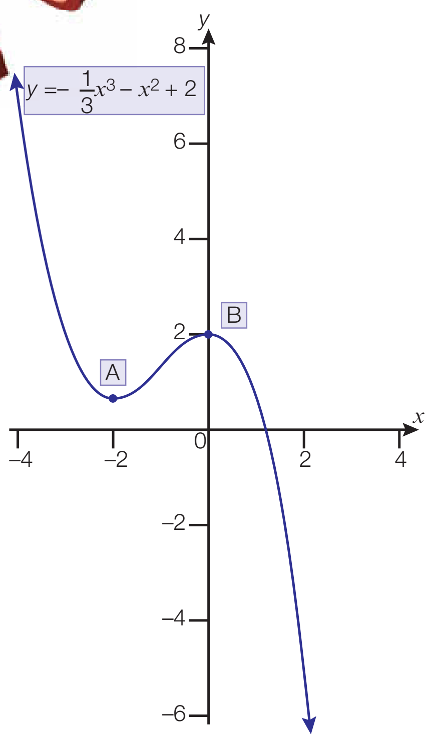
The CAPN navigation software (www.thecapn.com)

The graph shows the height of tides at Gloucester Harbor, Massachusetts, over a period of one day.

From midday to 6 p.m. the height of the water is increasing; high tide occurs at around 6 p.m., and after that the height of the water decreases until about midnight. The same pattern continues into the next day.

For any curve or function, you can give a similar description of which sections of it are 'increasing' and which sections are 'decreasing'.

Calculus allows you to add more detail and precision to such a description of functions.



Look at the graph of $f(x) = -\frac{1}{3}x^3 - x^2 + 2$.

The curve is decreasing from left to right up to the point A$(-2, \frac{2}{3})$. It is then increasing from A to B(0, 2), and on the right of B it is decreasing.

Using calculus, we can calculate the gradient at various points along the curve. Differentiating gives $f'(x) = -x^2 - 2x$, so:

$$f'(-3) = -(-3)^2 - 2(-3) = -3$$

$$f'(-1) = -(-1)^2 - 2(-1) = 1$$

$$f'(2) = -2^2 - 2 \times 2 = -8$$

RR ***You learned how to differentiate in Chapter 20.***

From these results, you can see that:

- At points where the function is **decreasing**, the gradient is negative: $f'(x) < 0$
- At points where the function is **increasing**, the gradient is positive: $f'(x) > 0$

A point where $f'(x) = 0$ is called a **stationary point**. The points A and B on the above graph are stationary points; the tangent to the curve at each of these points is a horizontal line, and the rate of change of y against x is instantaneously zero at those points. You can think of the curve as 'pausing' for an instant before changing direction. Another name for points like A and B is **turning points**.

Worked example 21.1

Q. Draw a graph of the function $f(x) = x^3 + 3x - 2$.

(a) Look at the graph and determine:

(i) where the function is increasing

(ii) where the function is decreasing.

(b) Confirm your results using calculus.

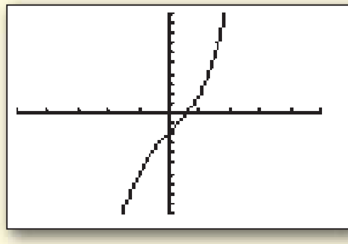
(c) Graph the function $g(x) = x^3 - 3x - 2$. How has the graph changed from that of $f(x)$? Why do you think this has happened?

continued . . .

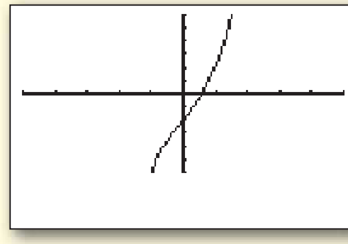
A. (a)

Use your GDC to draw the graph of $y = x^3 + 3x - 2$. (See '*224.2G Graphs*' on page 645 of the GDC chapter if you need to.)

TEXAS



CASIO



The curve seems to be sloping upwards throughout.

(i) The function is increasing at every point.

(ii) There is no part of the curve that is decreasing.

Differentiate $f(x)$ one term at a time using the formula on page 580. So for each term, multiply the term by the power, and reduce the power of x by 1.

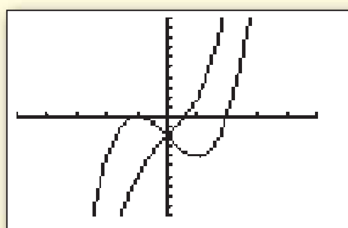
(b) $f(x) = x^3 + 3x - 2$

$f'(x) = 3x^2 + 3$

As $x^2 \geq 0$ for all values of x, it means that $3x^2 + 3 > 0$ for all values of x. This tells us that the gradient is always positive and hence there is no part of the curve that is decreasing.

(c)

TEXAS



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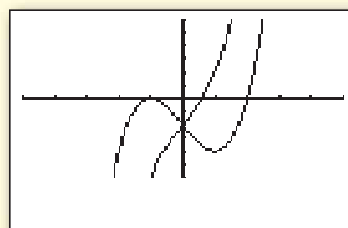


Use your GDC to draw the graph of $y = x^3 - 3x - 2$ on the same axes as the graph from part (a). Just by looking at the shape of the graph of $g(x)$ you can see that the gradient is negative in some places. Use the derivative $g'(x)$ to look for clues to the different shapes of the $f(x)$ and $g(x)$ curves.

$g'(x) = 3x^2 - 3$

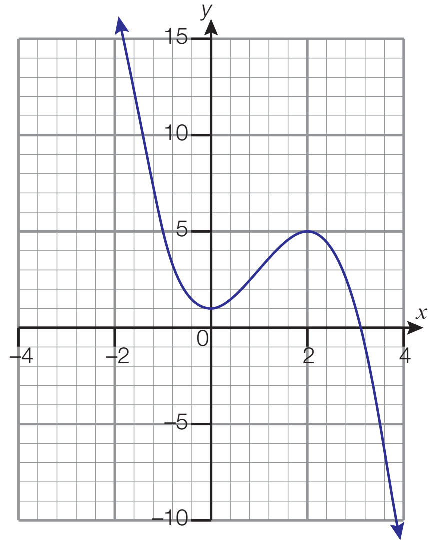
In contrast to $f(x)$, the gradient of $g(x)$ can be negative; this occurs for x between -1 and 1, so the graph is decreasing between the points $(-1, 0)$ and $(1, -4)$. This is confirmed by looking at the derivative $g'(x) = 3x^2 - 3$; although $x^2 \geq 0$ for all values of x, for some values of x subtracting 3 from $3x^2$ will lead to a negative value.

Exercise 21.1

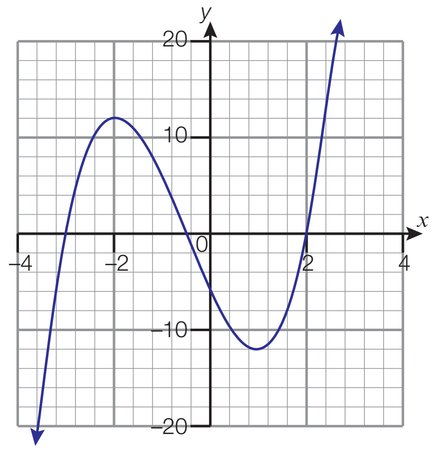
exam tip

Questions in IB examinations often ask for the values of x for which $f(x)$ is increasing or decreasing — the y value doesn't matter.

1. The graph of the function $f(x) = 1 + 3x^2 - x^3$ is shown. State the values of x between which the function is increasing and decreasing.



2. The graph of the curve $y = (2x + 1)(x - 2)(x + 3)$ is shown.



State the interval of x values for which the function is:

(a) increasing (b) decreasing.

3. Draw the graph of each of the following functions on your GDC and, by looking at the graph, determine the interval(s) of x values for which the function is:

(i) increasing (ii) decreasing

(a) $f(x) = 3x(x - 8)$ (b) $f(x) = x(2x + 9)$

(c) $y = x^2 - x - 56$ (d) $y = x^3 + 8x - 2$

(e) $y = x^3 - 12x + 3$ (f) $y = 2x^3 - 3x^2 - 12x + 6$

(g) $g(x) = \frac{x^3}{3} - 3x^2 + 5$ (h) $y = \frac{x^3}{3} + \frac{x^2}{2} - 4x + 1$

4. Find the range of values of x for which $f(x) = x^3 - 6x^2 + 3x + 10$ is an increasing function.

5. A function has equation $f(x) = 2 + 9x + 3x^2 - x^3$.

(a) Find $f'(x)$.

(b) Calculate $f'(-2)$.

(c) Determine whether $f(x)$ is increasing or decreasing at $x = -2$.

6. A curve is defined by the equation $f(x) = \frac{x^3}{3} - x^2 - 3x + 1$.

(a) Find $f'(x)$.

(b) Calculate:

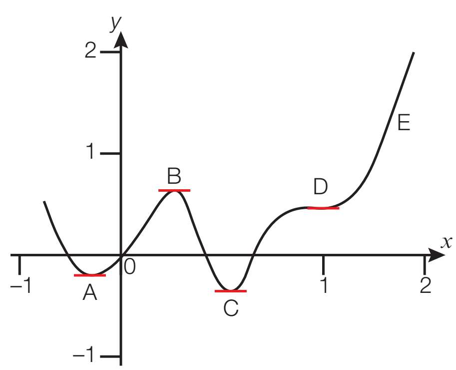
(i) $f'(-4)$ (ii) $f'(1)$.

(c) State whether the function is increasing or decreasing at:

(i) $x = -4$ (ii) $x = 1$.

21.2 Stationary points, maxima and minima

In the previous section, stationary points were defined as points where a curve has a gradient equal to zero. In this section, you will learn how to determine whether a stationary point is a maximum, minimum or inflexion point. A point of inflexion is a place where the curve 'pauses' but then continues in the same direction.



In the diagram, A, B, C and D are all stationary points.

- A, B and C are **turning points**, places where the curve changes direction.
- A and C are local **minimum points**, or local **minima** (plural of minimum), where the curve changes from decreasing to increasing.
- B is a local **maximum point** (plural: **maxima**), where the curve changes from increasing to decreasing.

Although a local maximum or local minimum is always a turning point, it is not necessarily the highest or lowest point on the entire graph; this is why the term 'local' is used. For instance, the point E has a greater y value than the local maximum B; but because the curve does not change direction at E, it is not a maximum.

exam tip

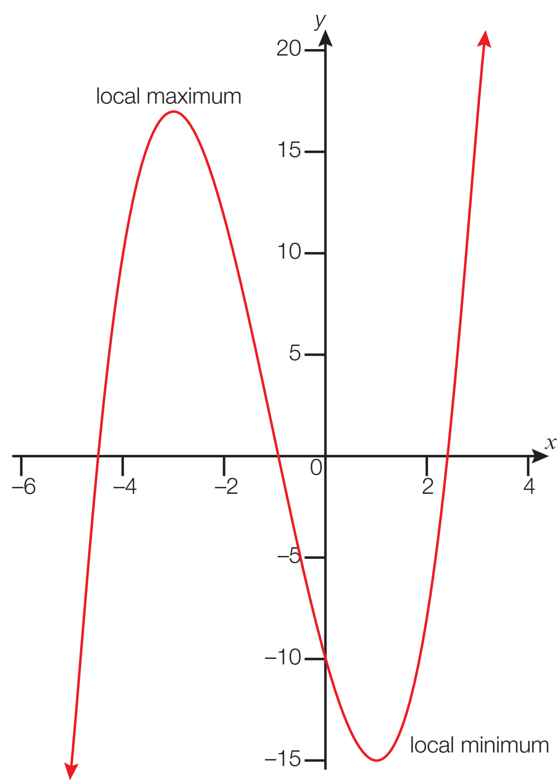
Points of inflexion are interesting to explore, but questions about them will not be set in examinations.

D is a point of **inflexion**; the curve is increasing before it reaches D, then it 'pauses' at this point and the gradient is instantaneously zero, and then it starts increasing again. There are also points of inflexion where the curve is decreasing on both sides. The important thing to remember is that inflexion points are stationary points which are **not** turning points.

To find the stationary points on a given curve, and to classify each of them as a local maximum or minimum, there are several steps that you need to work through.

For example, consider the function $f(x) = x^3 + 3x^2 - 9x - 10$.

1. It is a good idea to sketch the curve first. Using calculus, you can find stationary points without a diagram, but you are less likely to make mistakes if you start by drawing the curve on your GDC or computer.



2. Differentiate $f(x)$ to find the gradient function:

$$\frac{dy}{dx} = 3x^2 + 6x - 9$$

3. Stationary points occur where $\frac{dy}{dx} = 0$, so solve this equation to find the x-coordinates of any stationary points.

   $3x^2 + 6x - 9 = 0$

   From the GDC: $x = -3$ or $x = 1$

> This is the equation $\frac{dy}{dx} = 0$.
> This is a quadratic equation, so solve using one of the methods you met in Chapter 2.

4. Now that you know the x-coordinates of the stationary points, you can find the corresponding y-coordinates using the equation of the curve.

   $f(x) = x^3 + 3x^2 - 9x - 10$

   $f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 10 = -27 + 27 + 27 - 10 = 17$

   $f(1) = 1^3 + 3 \times 1^2 - 9 \times 1 - 10 = 1 + 3 - 9 - 10 = -15$

   So the stationary points of the curve are at $(-3, 17)$ and $(1, -15)$.

> Substitute $x = -3$ and $x = 1$ into the original equation.

5. To determine whether a stationary point is a maximum or a minimum (or neither), look at the gradient on either side of the stationary point.

   $f'(0.5) = 3(0.5)^2 + 6(0.5) - 9 = -5.25 < 0$

   $f'(1.5) = 3(1.5)^2 + 6(1.5) - 9 = 6.75 > 0$

> For the point $(1, -15)$, check the gradient at $x = 0.5$ and $x = 1.5$ by substituting these values of x into the equation of the derivative.

A table/diagram like the following makes it easy to see whether you have a maximum or a minimum.

Value of x	0.5	1	1.5
Value of $\frac{dy}{dx}$	-5.25	0	6.75
	$\searrow$	—	$\nearrow$

The point $(1, -15)$ is a local minimum because the curve changes from decreasing to increasing.

$f'(-3.5) = 3(-3.5)^2 + 6(-3.5) - 9 = 6.75 > 0$

$f'(-2.5) = 3(-2.5)^2 + 6(-2.5) - 9 = -5.25 < 0$

> For the point $(-3, 17)$, check the gradient at $x = -3.5$ and $x = -2.5$ by substituting these values of x into the equation of the derivative.

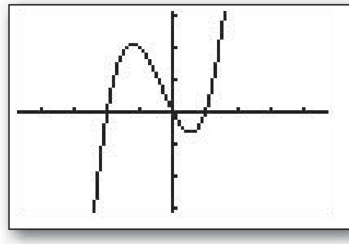
Make a similar diagram as before:

Value of x	-3.5	-3	-2.5
Value of $\frac{dy}{dx}$	6.75	0	-5.25
	$\nearrow$	—	$\searrow$

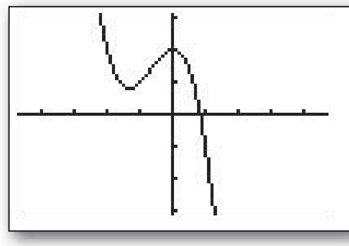
The point $(-3, 17)$ is a local maximum because the curve changes from increasing to decreasing.

For some types of function, you can use the shape of the general graph to help you decide whether a stationary point is a maximum or a minimum. For example,

- From left to right on the graph below, a cubic function whose x^3 term has positive sign will have a maximum followed by a minimum:



- From left to right on the graph below, a cubic function whose x^3 term has negative sign will have a minimum followed by a maximum.



So, for $f(x) = x^3 + 3x^2 - 9x - 10$, since the x^3 term has positive sign, you know immediately that the stationary point on the left, $(-3, 17)$, is a local maximum, while the stationary point on the right, $(1, -15)$, is a local minimum.

Summary

To find and classify the stationary points of a function:

1. Sketch the curve on your GDC.
2. Differentiate to find the gradient function.
3. Put $\frac{dy}{dx} = 0$ and solve the equation for the x-coordinate(s) of any stationary points.
4. Use the equation of the curve to find the y-coordinate for each stationary point.
5. Check the sign of $\frac{dy}{dx}$ on either side of each stationary point, to determine whether it is a local maximum or minimum.

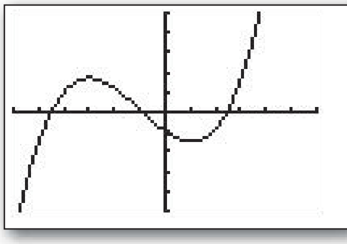
Using your GDC

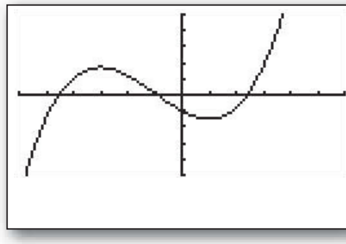
It is possible to use your calculator to a greater extent than in the example above to find and classify stationary points. See '*21.2 Finding local maximum and minimum points*' on page 690 of the GDC chapter for a reminder if you need to. However, to gain a better understanding of the mathematics, it is a good idea to practise the traditional calculus method described above before relying more heavily on your GDC.

Let's look at how you could use a GDC to find the stationary points of $f(x) = x^3 + 3x^2 - 9x - 10$.

TEXAS | CASIO

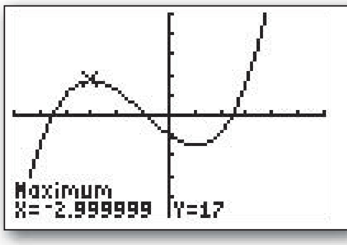
Enter the equation of the curve and draw the graph.

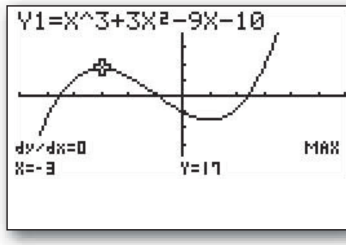




TEXAS | CASIO

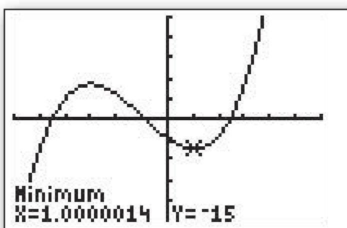
Find the maximum point. See '*21.2 Finding local maximum and minimum points*' on page 690 of the GDC chapter if you need to.

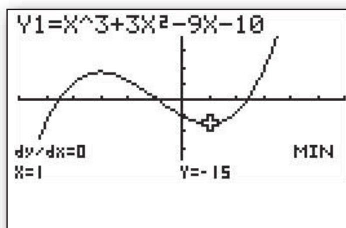




There is a maximum point at (–3, 17)

Find the minimum point.





There is a minimum point at (1, –15)

exam tip

Remember that your GDC cannot differentiate functions; so if a question asks you to find a derivative, that is a calculation you will **always** have to do for yourself.

Worked example 21.2

Q. Consider the function $f(x) = x^4 - x^3 - 12x + 1$.

(a) Find $f'(x)$.

(b) Evaluate $f(x)$ and $f'(x)$ at the point where $x = 2$. Is the function increasing or decreasing at this point?

(c) Calculate the coordinates of the turning points on the curve and determine the nature of the points.

continued . . .

Differentiate each term separately by multiplying the term by the power of x, then reducing the power of x by 1.

A. (a) $f'(x) = 4x^3 - 3x^2 - 12$

Using algebra and your GDC: for parts (b) and (c), we will show two ways of answering the question.

Substitute the value of $x = 2$ into the original function, and then into the derivative.

Method 1:

(b) When $x = 2$,

$f(2) = 2^4 - 2^3 - 12 \times 2 + 1 = -15$

$f'(2) = 4 \times 2^3 - 3 \times 2^2 - 12 = 8$

To see whether the function is increasing or decreasing, look at whether the gradient is positive or negative.

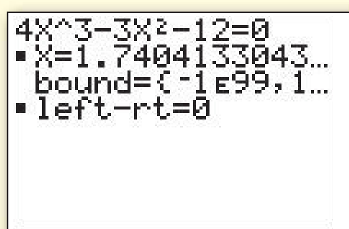
As $f'(2) > 0$, the function is increasing at the point $(2, -15)$.

At stationary points, $f'(x) = 0$.

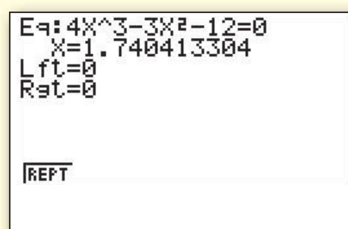
(c) $4x^3 - 3x^2 - 12 = 0$

This is a complicated equation to solve, so use your GDC. (See '*19.3 (b) Solving polynomial equations using an equation solver*' on page 685 of your GDC if you need to, and apply the principles to solving this cubic equation).

TEXAS



CASIO



There is only one solution, $x = 1.74$

Find the corresponding y value by substituting $x = 1.74$ into the original equation.

$f(x) = x^4 - x^3 - 12x + 1$

$f(1.74) = (1.74)^4 - (1.74)^3 - 12 \times 1.74 + 1$

When $x = 1.74$, $y = -16.0$ (3 s.f.), so there is a stationary point at $(1.74, -16.0)$.

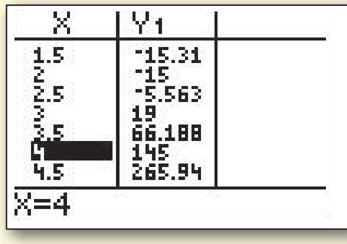
continued . . .

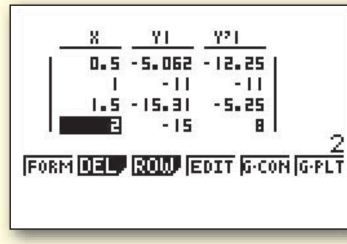
Value of x	1.5	1.74	2
Value of $f'(x)$	−5.25	0	8
	$\searrow$	—	$\nearrow$

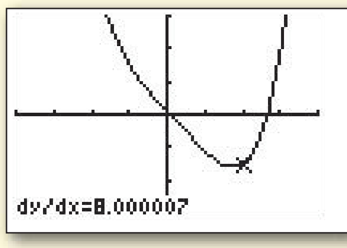
(1.74, −16.0) is a minimum point.

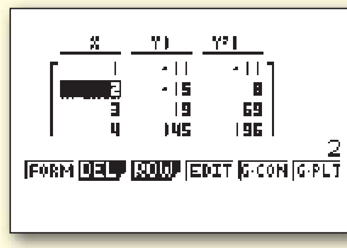
(b) Method 2: (*mostly your GDC*)

TEXAS **CASIO**









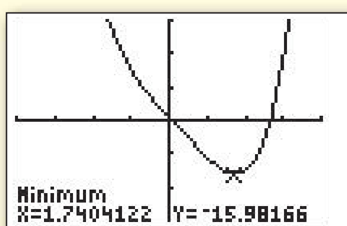
From the GDC:

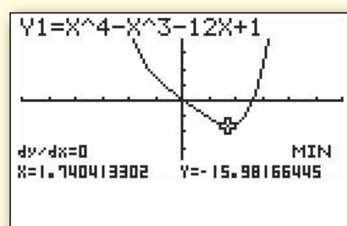
$f(2) = -15$, $f'(2) = 8$

As $f'(2) > 0$, the function is increasing at the point $(2, -15)$.

(c)

TEXAS **CASIO**





There is a minimum point at (1.74, −16.0).

To determine the nature of the stationary point, check the sign of the gradient on either side of it, and make a diagram to show this.

Using some algebra and your GDC: use the table of coordinates on your GDC to find the *y*-coordinate, and the table or graph to find the value of $\frac{dy}{dx}$. (If you need a reminder, see '*14.1 Accessing the table of coordinates from a plotted graph*' on page 678 and '*20.1 Finding the numerical value of the derivative*' on page 686 and '*21.1 Finding increasing and decreasing functions*' on page 689 of the GDC chapter.)

Use the GDC to locate the turning point. (See '*21.2 Finding local maximum and minimum points*' on page 690 of the GDC chapter if you need to.)

Exercise 21.2

1. For each of the following functions, draw the graph on your GDC and find all the stationary points on the curve, classifying them as minimum or maximum points.

(a) $y = x^2 + 4x$

(b) $y = 8x - x^2$

(c) $y = x^2 - 6x + 5$

(d) $f(x) = 4 + 3x - x^2$

(e) $g(x) = x^3 - x^2 - x$

(f) $h(x) = x^3 - 3x - 1$

(g) $y = 4 + 3x - x^3$

(h) $y = 4x^3 - 3x + 5$

(i) $y = x^3 - 4x^2 + 4x + 3$

(j) $f(x) = x^5 - 5x - 1$

2. Use calculus to find and classify the stationary points for all the functions in question 1. So, in each case:

- Find the gradient function.
- Equate the gradient function to zero and solve for the x values of stationary points.
- Find the corresponding y-coordinates.
- Determine the nature of the stationary points (i.e. whether they are maximum or minimum points).

3. A curve has equation $y = x^3 - 3x^2 - 8x - 11$.

(a) Find $\frac{dy}{dx}$.

The points P and Q are the stationary points on the curve.

(b) Find the coordinates of P and Q.

(c) Determine the nature of each of the stationary points.

4. A curve with equation $y = 2x^5 + 5x^2 - 3$ passes through the points R and S with coordinates $(-1, 0)$ and $(0, -3)$, respectively.

(a) Verify that the points R and S are stationary points on the curve.

(b) Determine the nature of the stationary points.

5. The equation of a curve is $y = x^3 - 4x$.

(a) Find $\frac{dy}{dx}$.

(b) Hence find the two values of x for which $\frac{dy}{dx} = 0$.

(c) Find the coordinates of the stationary points on the curve.

(d) Determine the nature of each stationary point.

6. If $f(x) = x^3 - 6x^2$, find the coordinates of the stationary points and determine whether they are maxima or minima.

7. Find the coordinates of the two stationary points on the curve with equation $y = x^3 - 3x$. Classify each of the points as a maximum or a minimum.

Learning links

21A Using the second derivative to classify stationary points

You can use the second derivative of a function to distinguish between local maxima and local minima.

- If $f''(x) > 0$ at a stationary point, then the stationary point is a local minimum.
- If $f''(x) < 0$ at a stationary point, then the stationary point is a local maximum.

Take the function $f(x) = x^3 + 3x^2 - 9x - 10$ that we investigated in section 21.2.

We saw that there are two stationary points, $(-3, 17)$ and $(1, -15)$.

Differentiating the derivative $f'(x) = 3x^2 + 6x - 9$ gives the second derivative:

$$f''(x) = 6x + 6$$

Now check:

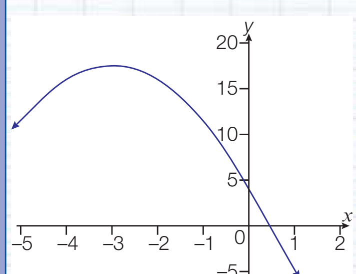
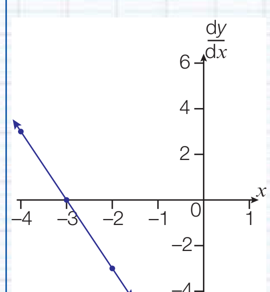
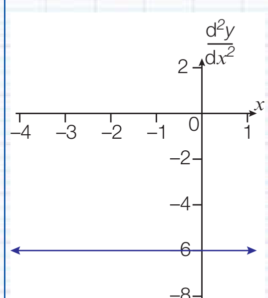
$$f''(-3) = 6 \times (-3) + 6 = -12$$

As $-12 < 0$, the point $(-3, 17)$ is a maximum.

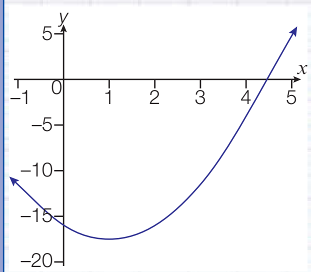
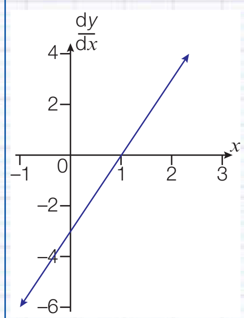
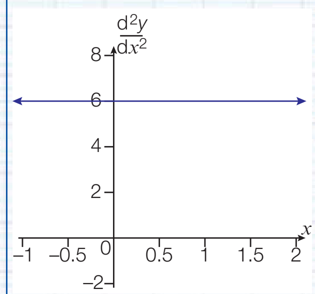
$$f''(1) = 6 \times 1 + 6 = 12$$

As $12 > 0$, the point $(1, -15)$ is a minimum.

The following table shows how the gradient of a function's graph changes around a maximum or minimum point, and what this means for the second derivative.

	$f(x)$ graph	$f'(x)$ graph	$f''(x)$ graph
Maximum point	$f(x) = -\frac{3}{2}x^2 - 9x + 4$  From left to right through the maximum, the curve changes from increasing to horizontal to decreasing.	 The gradient changes from positive to zero to negative (its graph slopes downward).	 The rate of change of the gradient, i.e. the second derivative, is negative: $f''(x) < 0$

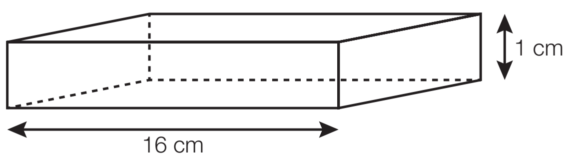
continued . . .

	$f(x)$ graph	$f'(x)$ graph	$f''(x)$ graph
Minimum point	$f(x) = \frac{3}{2}x^2 - 3x - 16$  From left to right through the minimum, the curve changes from decreasing to horizontal to increasing.	 The gradient changes from negative to zero to positive (its graph slopes upward).	 The rate of change of the gradient, i.e. the second derivative, is positive: $f''(x) > 0$

21.3 Optimisation

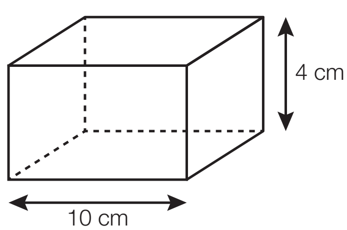
A group of friends are preparing for a party. Some of them are making candy boxes from 18 cm squares of coloured card. The boxes all have square bases but different depths. What is the volume of the boxes that they are making?

Zaira makes a shallow box, 1 cm deep, like this:



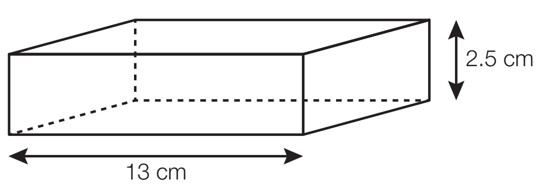
The volume of Zaira's box is $1 \times 16 \times 16 = 256\,\text{cm}^3$.

Mike makes a box that is deeper, with a depth of 4 cm:



The volume of Mike's box is $4 \times 10 \times 10 = 400\,\text{cm}^3$.

Polli makes her box like this; it has a volume of $422.5\,\text{cm}^3$.



Even though the size of the card used stays the same, the volume of the box seems to change depending on the depth of the box. How can the friends design a box made from an 18 cm by 18 cm piece of card with the greatest possible volume?

This is an **optimisation** problem, where you want to find the most efficient use of the resources that you have. In these circumstances we need to create a function to represent the situation and then plot the graph of this function to find the local maximum. In other examples, you might want to find the *smallest* value and you would plot a graph of the function and locate the local minimum.

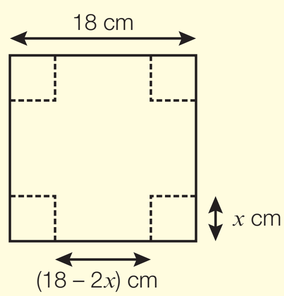
Worked example 21.3

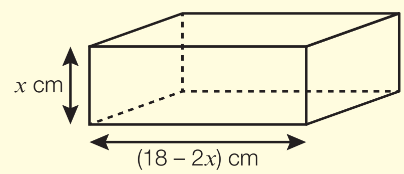
Q. How can the friends design a box made from an 18 cm by 18 cm piece of card with the greatest possible volume?

A.

To find the box with the largest volume, visualise the box as in the diagram, and then use algebra to generalise the problem.

By cutting out or folding an x cm square piece at each corner of the 18 cm square card, you can make a box that is x cm high, with a square base of side length $(18 - 2x)$ cm.





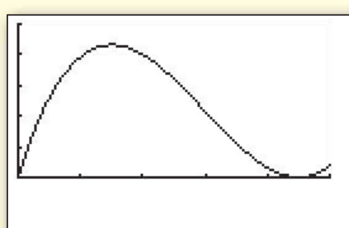
We know that the volume of a box = length × width × height.

In this case, the height (depth) is x cm and the length and width are both $(18 - 2x)$ cm.

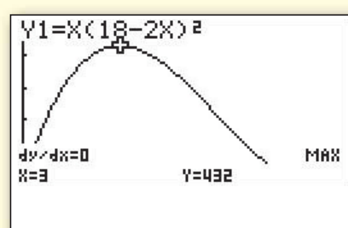
$$V = (18 - 2x)(18 - 2x)x$$

To find the value of x that will give the maximum volume, graph this equation on a GDC (see '*21.2 Finding local maximum and minimum points*' on page 690 of the GDC chapter if you need to.)

TEXAS



CASIO



There is a clear maximum point on the curve. Use your GDC to find the coordinates of the point.

continued . . .

From GDC: coordinate of local maximum is (3, 432). So, $x = 3$, $y = 432$. So the box with the greatest volume, made from an 18 cm square of card, is one that has a depth of 3 cm and a length and width of 12 cm, giving a volume of 432 cm^2.

Exercise 21.3

1. Sierra has a 20 cm by 20 cm square piece of card. She wants to make an open-top box out of the card. If she wants her box to have the maximum possible volume, find the dimensions of this box.

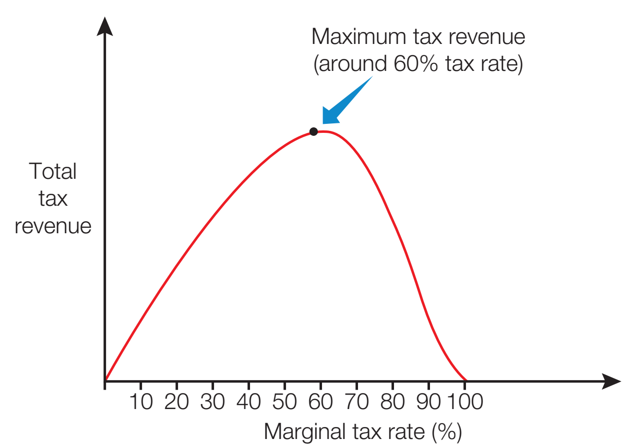
2. The sum of two positive integers is 13. Find the maximum product you can get from the pair of integers.

3. A rectangle has a perimeter of 24 cm. Calculate the dimensions of the rectangle which will result in the maximum area. You may assume that the lengths of the sides of the rectangle can be integer values only.

4. The area of a rectangle is 18 cm^2. Calculate the dimensions which will result in the minimum perimeter. You may assume that the lengths of the sides of the rectangle can be integer values only.

Using calculus to solve optimisation problems

Problems asking for maximum or minimum solutions occur in many different contexts. For instance, the Laffer curve, first proposed by the American economics professor Arthur Laffer, suggests that there is a maximum amount of tax that can be imposed on the citizens of a country. If a government asks its citizens for more than 60% of their income in tax, the government will lose revenue rather than gain it.



- In business, it is useful to be able to predict whether you can make more profit by selling many things cheaply or fewer things at a higher price.
- In manufacturing, optimisation can help a company find the break-even price of an item, that is, the point where production costs and revenue would be equal.
- Packaging companies need to calculate the most economical dimensions for different shapes and styles of package.
- Doctors want to find out the most effective dosage for a medicine, and when the drug would be at its highest concentration in a patient's bloodstream.

Problems like these can all be solved by:

- finding an equation that describes the problem
- using calculus to look for a maximum or minimum point.

To solve an optimisation problem, work through the following steps:

1. Read the question carefully and make sure you understand it. You may need to read it more than once to absorb all the information.

2. Draw a diagram if possible.

3. Formulate an equation that links the variables of interest. With the techniques we have learned so far, you can use only two variables.

4. Differentiate the equation to find the gradient function.

5. Set the gradient function equal to zero to find the stationary points.

6. Solve the equation for the stationary points and check whether you have a stationary point of the type you are seeking (maximum or minimum).

7. Use your results to give your answer to the original problem.

hint

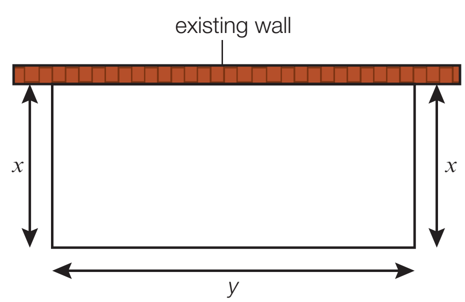
Not every step will be needed for every problem, but the sequence of steps is always the same.

Let's use the steps above to solve the following problem.

A zoo needs a rectangular enclosure for some small animals. The enclosure can use one wall of an existing building, and there are 80 metres of fencing panels available for the remaining three sides. Find the maximum area of the enclosure and its dimensions.

1. Read the question. Why does it mention the 'remaining three sides'?

2. Draw a diagram and mark in the information from the question:



3. The question asks for an area. The area of the enclosure is $A = x \times y$.

   This equation has three variables, A, x and y; however, the question gives you enough information to make a link between x and y, which would allow the equation to be simplified.

   Since there is 80 m of fencing available, the diagram shows that $x + y + x = 80$. So you can write $y = 80 - 2x$, and now,

$$\begin{aligned} A &= x \times (80 - 2x) \\ &= 80x - 2x^2 \end{aligned}$$

4. Differentiating gives $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 80 - 4x$.

5. For stationary points, $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$, that is, $80 - 4x = 0$.

6. Solving $80 - 4x = 0$ gives $x = 20$.

   If $x = 20$, then $y = 80 - 2 \times 20 = 40$.

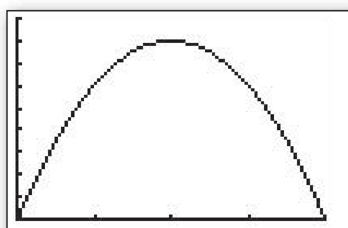
7. The maximum area of the enclosure is $40 \times 20 = 800\,\text{m}^2$, and its dimensions are $40\,\text{m} \times 20\,\text{m}$.

Using your GDC, you could follow steps 1 to 3 as before, and then graph the equation $A = 80x - 2x^2$ on your GDC and find the coordinates of the maximum point directly:

GDC

See '*21.2 Finding local maximum and minimum points*' on page 690 of the GDC chapter if you need a reminder.

TEXAS



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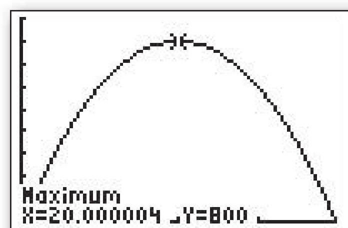


Worked example 21.4

Q. Myra and Salim (from Worked example 20.8) are looking again at their Business Studies project of scarf manufacturing. They now calculate that their fixed costs are \$25, which will remain the same regardless of how many scarves they make. The material for each scarf costs \$8, but this will decrease by \$0.20 for each additional item made, as their supplier will give them a discount if they buy more material. At what production level will the total costs reach a maximum, and what is this maximum cost?

continued . . .

Total cost = fixed costs + variable costs, where the variable costs depend on the number of items produced.

A. Let x be the number of scarves made; then

$C(x) = 25 + x(8 - 0.2x)$

$C(x) = 25 + 8x - 0.2x^2$

Differentiate to find the rate of change of total cost with respect to number of items.

$C'(x) = 8 - 0.4x$

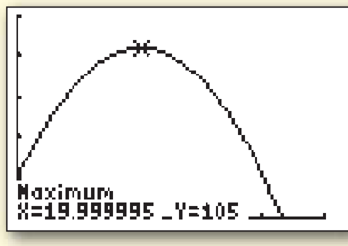
Solve the equation $\frac{dC}{dx} = 0$ to find stationary points.

$8 - 0.4x = 0$

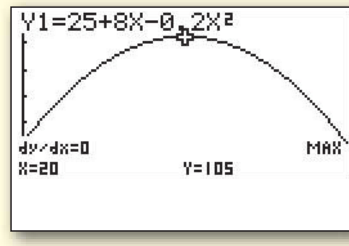
$x = 20$

Check that the stationary point is a maximum, and find its *y*-coordinate. You can use the GDC for this. (See '*21.2 Finding local maximum and minimum points*' on page 690 of the GDC chapter if you need to.)

TEXAS



CASIO



When $x = 20$, $C(x) = 105$.

The maximum cost, \$105, occurs when they make 20 scarves.

Worked example 21.5

Q. The formula for the rate P at which a car's engine is working is:

$$P = 12v + \frac{5000}{v}$$

where v is the speed of the car in m s^{-1}.

(a) Find $\frac{dP}{dv}$.

(b) Use your expression in (a) to calculate the speed of the car when the engine is working at its most efficient.

(c) Confirm your result in (b) with a graph.

continued . . .

Rewrite the equation as a sum of powers of *v*.

A. (a) $P = 12v + 5000v^{-1}$

Differentiate with respect to *v*.

$$\frac{dP}{dv} = 12 - 5000v^{-2} = 12 - \frac{5000}{v^2}$$

'Most efficient' tells you that this is an optimisation problem, so look for stationary points.

(b) $\frac{dP}{dv} = 0$

$$12 - \frac{5000}{v^2} = 0$$

$$12 = \frac{5000}{v^2}$$

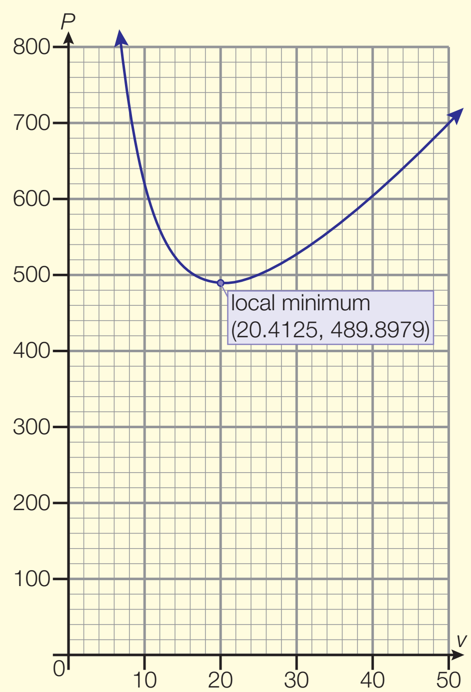
Use your GDC to solve the equation, (see '*19.2 Solving unfamiliar equations*' on page 684 of the GDC chapter if you need to.)

From GDC: $v = 20.4$

The engine is at its most efficient when the car's speed is $20.4\,\mathrm{m\,s^{-1}}$.

Drawing the graph confirms that $v = 20.4$ is a local minimum; that is, the engine uses the least effort at this speed.

(c)



Exercise 21.4

1. The profit function of a certain manufacturing company is:

$$P = 25 + 120q - 4q^2,$$

where P is the profit in thousands of pounds and q is the output level in number of units produced.

(a) Find $\dfrac{dP}{dq}$.

(b) Determine the value of q for which $\dfrac{dP}{dq} = 0$.

(c) Hence find the maximum profit, P_{max}.

2. The total cost function of a certain manufacturing company is given by:

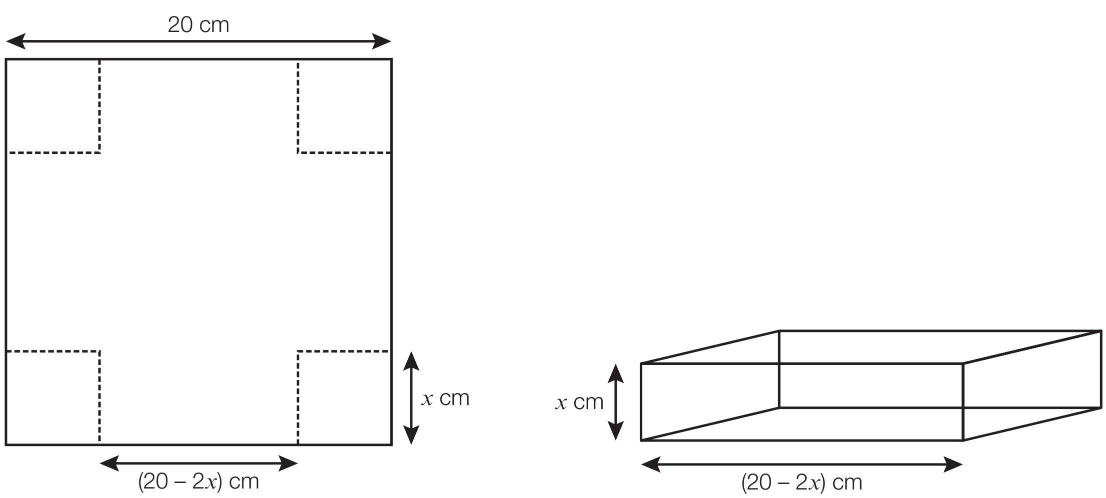
$$C = 3800 - 240n + 1.5n^2$$

where C is the total cost in thousands of dollars and n is the number of items produced.

(a) Determine the value of n which minimises the total cost.

(b) Use calculus to justify your reason for deciding that your answer is the minimum rather than the maximum value.

3. The diagram shows a 20 cm by 20 cm square piece of card. A square of side x cm is cut from each corner of the card to make an open box with a square base and a height of x cm.



(a) Show that the volume, V cm^3, of the box can be written as

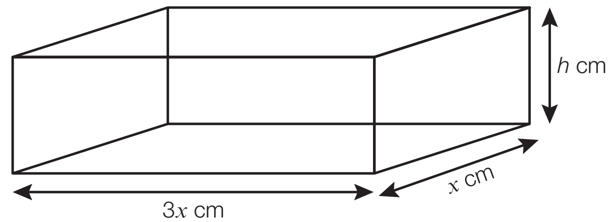
$$V = 4x^3 - 80x^2 + 400x$$

(b) Find $\dfrac{dV}{dx}$.

(c) Determine the value of x for which the volume of the box is maximum.

(d) Hence find the dimensions of the box that give the maximum volume. Calculate the corresponding volume.

4. Melissa is designing an open-topped toy box out of cardboard. The base of the box is rectangular with length three times as long as the width. The total surface area of the five faces of the box is 7488 cm^2.



(a) Taking the width of the box to be x cm and the height to be h cm, show that:

$$3x^2 + 8xh = 7488$$

(b) By expressing h in terms of x, show that the volume, V, of the box can be written as:

$$V = 2808x - \frac{9x^3}{8}$$

(c) Find $\frac{dV}{dx}$.

(d) Hence determine the value of x corresponding to the maximum volume of the box.

(e) Find the dimensions of the box that give the maximum volume.

(f) State the maximum volume of the box.

5. The total revenue function of a company is:

$$R = 320n - 4n^2$$

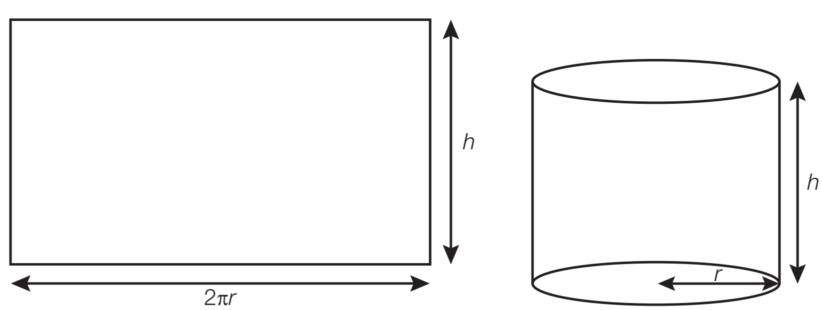
where R is the total revenue in thousands of dollars and n is the number of items produced.

(a) Find $\frac{dR}{dn}$.

(b) Hence find the value of n which maximises the total revenue. Justify your reason for deciding it is the maximum.

(c) Calculate the maximum total revenue.

6. Josephine has bought a piece of rectangular card with a **perimeter** of 120 cm. She wants to roll the rectangle into a cylinder with the largest possible volume.



(a) If she labels the piece of card as shown in the diagram, show that $h + 2\pi r = 60$.

(b) The volume of a cylinder is given by $V = \pi r^2 h$. Use your result from (a) to show that V can also be written as $V = \pi r^2(60 - 2\pi r)$.

(c) Find $\frac{dV}{dr}$.

(d) Calculate the value of r that will give Josephine the greatest volume for her cylinder.

(e) Find the volume of this largest cylinder.

Summary

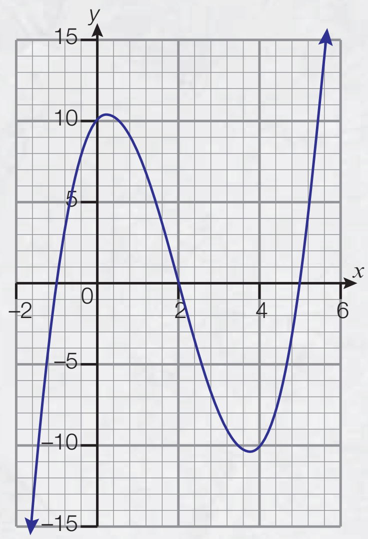
You should know:

- how to identify increasing and decreasing functions
- how to interpret graphically the gradient $f'(x)$
  - at points where the function is decreasing, the gradient is negative, $f'(x) < 0$
  - at points where the function is increasing, the gradient is positive, $f'(x) > 0$
  - at points where the gradient is equal to zero, $f'(x) = 0$, the point is a stationary point
- how to find values of x where the gradient of a curve is zero
- that a stationary point can be
  - a turning point
    - that is a local minimum (the curve changes from decreasing to increasing)
    - that is a local maximum (the curve changes from increasing to decreasing)
  - a point of inflexion (where the curve continues in the same direction, i.e. points of inflexion are stationary points that are **not** turning points)
- how to find a stationary point and identify what type of stationary point it is
- that optimisation problems require you to construct an equation, the local maximum or minimum of which is the solution to the optimisation problem
- that the solution of the equation $f'(x) = 0$ is important because you can use it to solve optimisation problems by finding a local maximum or minimum
- how to use calculus to solve optimisation problems that involve maximising or minimising a certain quantity.

Mixed examination practice

Exam-style questions

1. The following graph shows the function with equation $y = x^3 - 6x^2 + 3x + 10$.



State the values of x for which the function is:

(a) increasing

(b) decreasing.

2. A function is defined as $f(x) = 7x^3 - 12x + 3$.

(a) Find $f'(x)$.

(b) Calculate $f'(4)$.

(c) Determine whether $f(x)$ is increasing or decreasing at $x = 4$.

3. The curve C with equation $y = 3 + 6x^2 - 4x^3$ passes through the points P(0, 3) and Q(1, 5).

(a) Find $\frac{dy}{dx}$.

(b) Verify that the points P and Q are stationary points on the curve.

(c) Determine the nature of each stationary point.

4. The equation of a curve is $y = 2x^3 - 9x^2 - 24x + 3$.

(a) Find $\frac{dy}{dx}$.

(b) Hence find the two values of x for which $\frac{dy}{dx} = 0$.

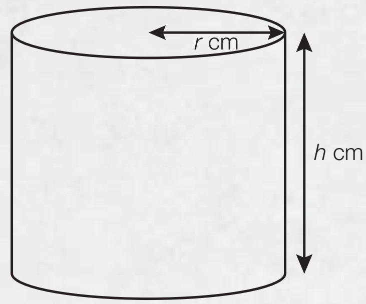
(c) Find the coordinates of the stationary points on the curve.

(d) Determine the nature of each stationary point.

5. The equation of a curve is given by $f(x) = 5 + 6x^2 - x^3$.
The points R and S on the curve are the minimum and maximum points, respectively.
Find the coordinates of R and S. Justify your answer using calculus.

6. Francis the farmer plans to section off part of a field to form a rectangular enclosure. He has 200 m of fencing material. The enclosure will use one wall of an existing building, and the remaining three sides will be fenced. Find the maximum area of the enclosure and its corresponding dimensions.

7. Liam is designing a drinks can for his Technology project. The cylindrical can should hold 330 cm^3 of fluid. Liam wants to minimise the material needed for producing the can.



Use calculus to determine the dimensions of the can (radius r and height h) that will minimise the surface area.

8. A farmer has 800 m of fencing material. Determine the dimensions of the rectangular enclosure that will maximise the fenced area. Work out the maximum area of the enclosure.

9. A box with a square base is to be designed to have a volume of 8000 cm^3.

(a) Find the dimensions of the box which will minimise the amount of material used.

(b) Determine the minimum surface area of the box.

10. Repeat question 9 for an open-topped, square-based box of the same volume.

Past paper questions

1. A function is represented by the equation:

$$f(x) = ax^2 + \frac{4}{x} - 3$$

(a) Find $f'(x)$. *[3 marks]*

The function $f(x)$ has a local maximum at the point where $x = -1$.

(b) Find the value of a. *[3 marks]*

[Total 6 marks]

[Nov 2007, Paper 1, Question 15] (© *IB Organization 2007*)

2. A football is kicked from a point A$(a, 0)$, $0 < a < 10$, on the ground towards a goal to the right of A.

The ball follows a path that can be modelled by **part** of the graph:

$$y = -0.021x^2 + 1.245x - 6.01,\ x \in \mathbb{R},\ y \geq 0.$$

x is the horizontal distance of the ball from the origin
y is the height above the ground
Both x and y are measured in metres.

(a) Using your graphic display calculator or otherwise, find the value of a. *[1 mark]*

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[2 marks]*

(c) (i) Use your answer to part (b) to calculate the horizontal distance the ball has travelled from A when its height is a maximum.

(ii) Find the maximum vertical height reached by the football. *[4 marks]*

(d) Draw a graph showing the path of the football from the point where it is kicked to the point where it hits the ground again. Use 1 cm to represent 5 m on the horizontal axis and 1 cm to represent 2 m on the vertical scale. *[4 marks]*

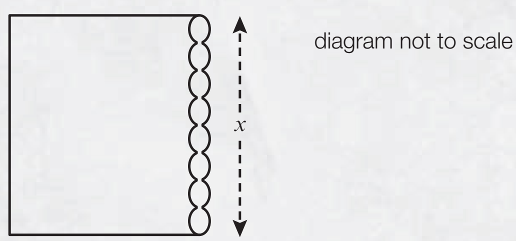
The goal posts are 35 m from **the point where the ball is kicked**.

(e) At what height does the ball pass over the goal posts? *[2 marks]*

[Total 13 marks]

[May 2007, Paper 2, Question 3 (ii)] (*© IB Organization 2007*)

3. A farmer has a rectangular enclosure with a straight hedge running down one side. The area of the enclosure is $162\ \mathrm{m}^2$. He encloses this area using x metres of the hedge on one side as shown on the diagram below.



(a) If he uses y metres of fencing to complete the enclosure, show that $y = x + \frac{324}{x}$. *[3 marks]*

The farmer wishes to use the least amount of fencing.

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[3 marks]*

(c) Find the value of x which makes y a minimum. *[3 marks]*

(d) Calculate this minimum value of y. *[2 marks]*

(e) Using $y = x + \frac{324}{x}$ find the values of a and b in the following table.

x	6	9	12	18	24	27	36
y	60	45	39	a	37.5	b	45

[2 marks]

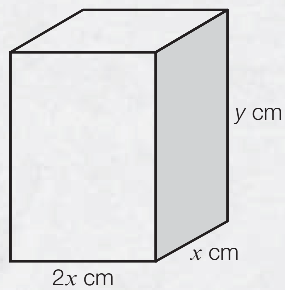
(f) Draw an accurate graph of this function using a horizontal scale starting at 0 and taking 2 cm to represent 10 metres, and a vertical scale **starting at 30** with 4 cm to represent 10 metres. *[5 marks]*

(g) Write down the values of x for which y increases. *[2 marks]*

[Total 20 marks]

[Nov 2006, Paper 2, Question 5] (© *IB Organization 2006*)

4. A closed rectangular box has a height y cm and width x cm. Its length is twice its width. It has a fixed outer surface area of 300 cm^2.



(a) Show that $4x^2 + 6xy = 300$. *[2 marks]*

(b) Find an expression for y in terms of x. *[2 marks]*

(c) Hence show that the volume V of the box is given by $V = 100x - \frac{4}{3}x^3$. *[2 marks]*

(d) Find $\frac{dV}{dx}$. *[2 marks]*

(e) (i) Hence find the value of x and of y required to make the volume of the box a maximum.

(ii) Calculate the maximum volume. *[5 marks]*

[Total 13 marks]

[May 2008, Paper 2, TZ1, Question 5(ii)] (© *IB Organization 2008*)

5. The function $f(x)$ is such that $f'(x) < 0$ for $1 < x < 4$. At the point P(4, 2) on the graph of $f(x)$ the gradient is zero.

(a) Write down the equation of the tangent to the graph of $f(x)$ at P. *[2 marks]*

(b) State whether $f(4)$ is greater than, equal to or less than $f(2)$. *[2 marks]*

(c) Given that $f(x)$ is increasing for $4 \leq x < 7$, what can you say about the point P? *[2 marks]*

[Total 6 marks]

[May 2008, Paper 1, TZ2, Question 15] (© *IB Organization 2008*)