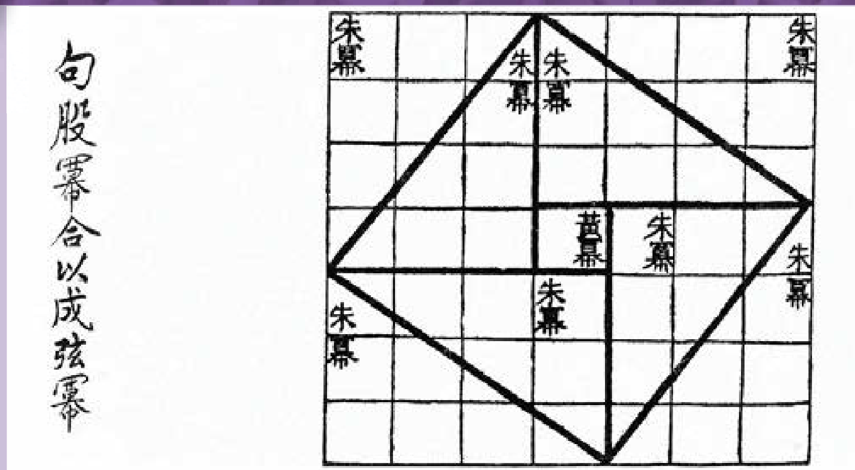
5 Geometry and trigonometry



An ancient Chinese diagram illustrating Pythagoras' theorem.

The study of geometry, and in particular the mathematics of right-angled triangles, goes back to the earliest civilisations.

Every time that you use Pythagoras' theorem, or solve a triangle using trigonometry, imagine yourself back with a student sitting on a mat in Babylon, or puzzling out the answer in ancient China, India or Egypt.

Pythagoras' theorem is named after Pythagoras of Samos (~ 569–475 BCE), a Greek who worked in the eastern Mediterranean and southern Italy. However, his particular proof is just one amongst many.

A Chinese textbook (*Zhou Bi Suan Jing*) contains a description of the same theorem and examples of how to apply it to practical problems. This dates from an era of about 300 BCE.

There are early sets of **Pythagorean triples** on a clay tablet from Babylon that dates from ~1800 BCE, and the theorem is well documented in early Indian sources. Builders in Egypt, in the time of the Pharaohs, used a rope stretcher based on the 3,4,5 triangle to create a right angle.

Trigonometry was probably developed to support the study of astronomy. The prediction of eclipses was always important, as religious ceremonies had to be timed to coincide with them. The earliest documented evidence of the use of the sine is about 500 CE, in India in the time of Aryabhata I.

Did all these scholars have the same ideas in isolation? Or did they communicate with each other and pass theories and techniques along trade routes and seaways? Was the mathematics they used waiting for them to discover it? Or did they all invent it and then tell each other about it? Historians and archaeologists know that there was much more communication in those early times than previously thought. Does this explain the exchange of ideas more than 2000 years ago?

Prior learning topics

It will be easier to study this topic if you:

- know the basic geometric concepts: point, line, plane and angle
- recognise simple two-dimensional shapes such as circles, triangles, quadrilaterals and combinations of these
- know how to calculate the areas and perimeters of simple and compound shapes
- know Pythagoras' theorem and how to use it
- know how to plot coordinates in two dimensions
- know how to find the midpoints of lines on a graph and the distance between two points on a graph.

Chapter 14 Equation of a line in two dimensions

In this chapter you will learn:

- about the gradient (m) of a line and how to calculate it
- how to identify parallel lines ($m_1 = m_2$) and perpendicular lines ($m_1 \times m_2 = -1$)
- about the y-intercept (c) of a line
- more about the equation of a line in two dimensions given in either form: $y = mx + c$ or $ax + by + d = 0$

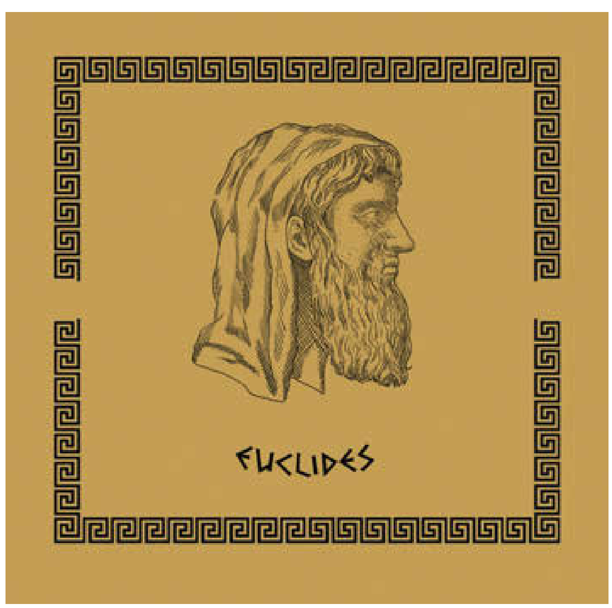
Euclid (~300 BCE) was a Greek mathematician who worked in Alexandria in Egypt. Whilst leading the mathematical school in Athens, he wrote a textbook in 13 parts called *The Elements* in which he collected together all the mathematics known at that time. Euclid's reputation is founded on the way he collated, and improved, theorems and ideas that were current then, which was mainly geometry. *The Elements* was the standard textbook for mathematicians until about 150 years ago. Euclid can be described as the most important mathematics teacher of all time.

The Elements begins with definitions and five postulates, or axioms. The first postulate states that it is possible to draw a straight line between any two points. This may seem obvious to you as you have drawn lines on pieces of paper since you were young. But consider that any straight line is constructed from an infinite number of points on that line and to draw a straight line on a flat piece of paper, you only need two pieces of information. With those two pieces of information you can draw a line that is unique.

You need:

- either a point on the line and its direction or gradient,
- or two points on the line.

This was groundbreaking at the time but is something that we take for granted today, and students often question why we need to learn it. There is a story about someone who had begun to learn geometry with Euclid who wondered a similar thing. After he had learned the first theorem, the student asked Euclid, 'What shall I get by learning these things?' Euclid responded by asking his slave to, 'Give him three pence since he must make gain out of what he learns.'

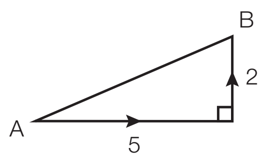


14.1 The gradient of a line

The **gradient** of a line describes the 'steepness' or slope of that line. A very steep line has a large gradient, while a shallow slope has a small gradient.

$$\text{The gradient is defined as: } m = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}}$$

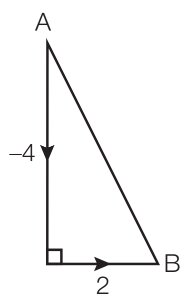
m is the letter that represents the gradient of a line in the general linear equation $y = mx + c$.



The vertical change from A to B is +2.

The horizontal change from A to B is +5.

$m = \frac{2}{5}$



The vertical change from A to B is –4.

The horizontal change from A to B is +2.

$m = \frac{-4}{2} = -2$

A line that 'slopes downwards' has a negative gradient.

Calculating the gradient from the coordinates of the points on the line

If a line is plotted on a graph, the gradient can be calculated using the **coordinates** of points on that line. Be careful though and look at the scale of the graph, both horizontally and vertically. In the graph below, two units on the horizontal axis is the same as two units on the vertical axis, but this might not always be the case.

hint

A Cartesian plane (below) is a system of two axes on a flat expanse. It is named after René Descartes, who originated it.

AC is 2 units long,

BC is 4 units long.

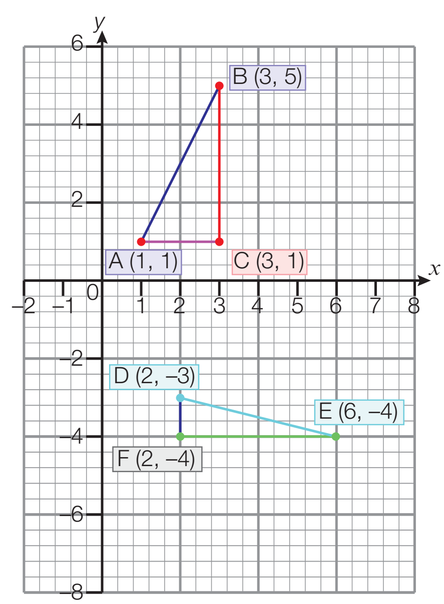
$m_{AB} = \frac{4}{2} = 2$

DF is 1 unit long,

EF is 4 units long.

$m_{ED} = \frac{-1}{4} = -\frac{1}{4}$

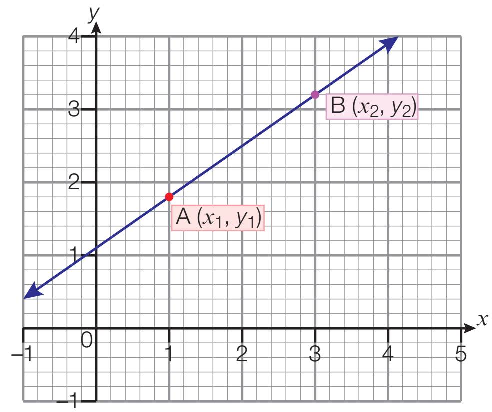
Note that the line DE slopes downwards so it has a negative gradient. F is –1 along the y-axis from D.



hint

The suffixes to x and y must be in the same order. This will ensure that the gradient has the correct sign.

Using the general coordinates (x_1, y_1) and (x_2, y_2), the vertical distance from A to B in the graph below is $y_2 - y_1$ and the horizontal distance from A to B is $x_2 - x_1$.



The gradient can be calculated as:

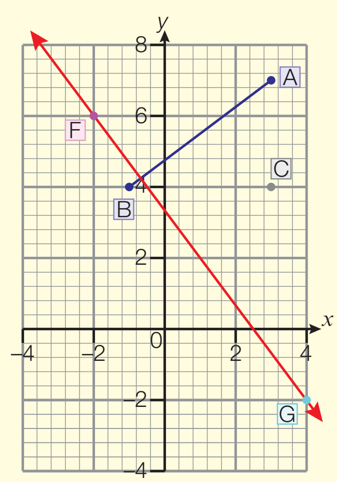
$$m = \frac{\text{change in vertical distance}}{\text{change in horizontal distance}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Therefore, if you are given the coordinates of two points on a line, you can calculate its gradient in one of two ways:

1. Draw a graph, plot the points and calculate the vertical and horizontal changes.
2. Use the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

The calculations of gradients can be important in civil engineering. The trans-Canada highway is engineered so that gradients in the road (any inclines or declines) are kept at about 6%. This helps to regulate traffic speed and flow.

Worked example 14.1



Q. (a) Find the gradient of the line joining the points A(3, 7) and B(−1, 4).

continued . . .

(b) A line passes through the points F(−2, 6) and G(4, −2). Calculate the gradient of the **line segment** FG.

(c) C is the point (3, 4). Calculate the gradients of the lines AC and BC.

Substitute the coordinates into the formula for the gradient: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

A. (a) $m_{AB} = \frac{7-4}{(3-(-1))} = \frac{3}{4}$

$m = \frac{3}{4}$

Notice that FG 'slopes backwards' so this gradient is negative.

(b) $m_{FG} = \frac{6-(-2)}{(-2)-4} = \frac{8}{-6} = -\frac{4}{3}$

$m = -\frac{4}{3}$

When trying to divide by zero, your calculator may return the answer, 'Syntax ERROR' or 'Ma ERROR'.

(c) $m_{AC} = \frac{7-4}{3-3} = \frac{3}{0}$.

AC is a vertical line and so has an undefined gradient.

This is incalculable. The gradient of a vertical line is undefined. The symbol ∞ is used to denote this.

The gradient of any horizontal line is zero.

$m_{BC} = \frac{4-4}{3-(-1)} = \frac{0}{4} = 0$

BC is a horizontal line.

The next worked example shows how to calculate gradients of parallel lines and of lines that are **perpendicular** to each other.

Worked example 14.2

Q. (a) Plot the points A(1, 1), B(3, 5), C(2, 4), D(8, 1), E(−4, −5) and F(−1, 1) on a graph and join the points A to B, C to D and E to F.

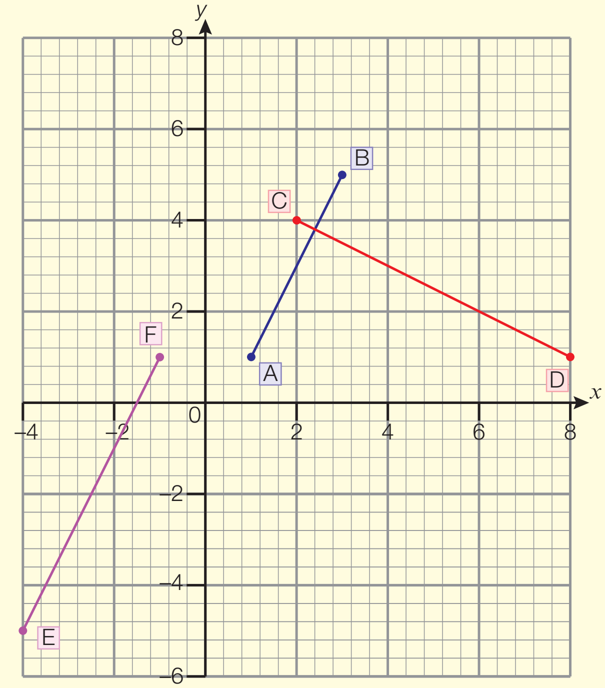
(b) Calculate the gradients of the line segments AB, CD and EF.

(c) Which two lines are parallel? Look at their gradients and comment.

(d) Which two lines are perpendicular?

continued . . .

A. (a)



(b) $m_{AB} = \frac{4}{2} = 2$

$m_{CD} = \frac{3}{-6} = -\frac{1}{2}$

$m_{EF} = \frac{6}{3} = 2$

(c) AB and EF are parallel; they have the same gradient.

Remember that lines are parallel if they go in the same direction but never touch; the shortest distance between them is constant. You can see that AB and EF are parallel.

(d) AB and CD are perpendicular.

$m_{AB} \times m_{CD} = 2 \times -\frac{1}{2} = -1$

EF and CD are also perpendicular.

Two lines are perpendicular if they form a right angle where they intersect. If $m_1 \times m_2 = -1$ then the lines are perpendicular.

Gradient summary

- Horizontal lines have a gradient of 0.
- Vertical lines have a gradient that is undefined.
- Parallel lines have the same gradient, $m_1 = m_2$.
- The gradients of two lines that are perpendicular multiply to give −1: $m_1 \times m_2 = -1$.

Exercise 14.1

1. Plot the following sets of points and hence calculate the gradient of each of the lines joining the points.

   (a) (2, 0) and (0, 2)

   (b) (1, 7) and (3, 15)

   (c) (−2, 3) and (6, −8)

   (d) (−4, −5) and (0, 7)

   (e) (8, −9) and (12, 10)

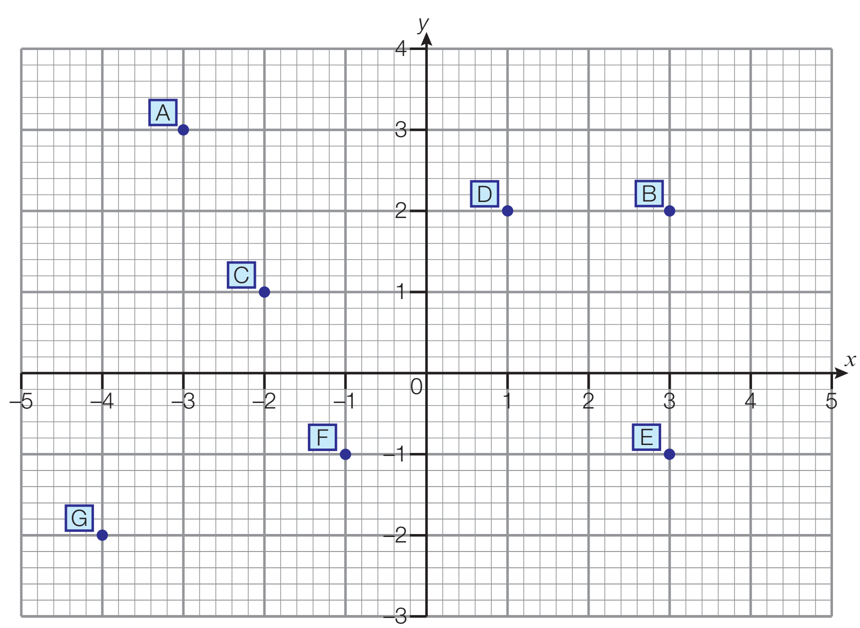
2. Plot the following points on a graph: A(−1, 2), B(1, 3), C(5, 0) and D(1, −2). Join the points to form a quadrilateral ABCD.

   (a) Calculate the gradients of the line segments AB, BC, CD and AD.

   (b) Which two of the line segments are parallel? Look at their gradients. Comment on the connection between their gradients.

   (c) Name a pair of perpendicular line segments. Look at their gradients. Comment on the connection between their gradients.

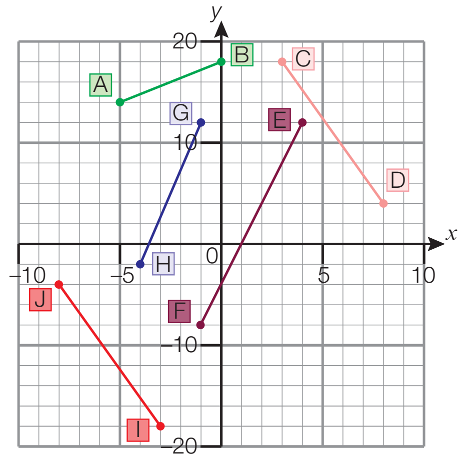
3. (a) Draw the line segments AE, CD, CE, DE, DF and GF on a copy of this grid. Hence calculate the gradient of each of them.



   (b) Which of the line segments from above are parallel? Comment on the relationship between their gradients.

   (c) Which of the line segments from above are perpendicular? Comment on the relationship between their gradients.

4. (a) Calculate the gradients of each of the following line segments.



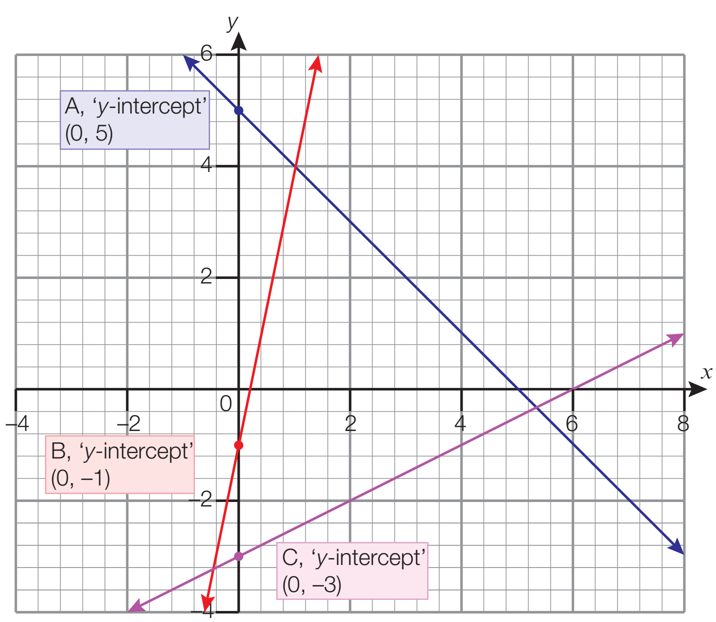
(b) Indicate, with reasons, which two of the line segments are:

(i) parallel

(ii) perpendicular.

14.2 The *y*-intercept

The point where a line crosses the *y*-axis is called the ***y*-intercept**. This is often denoted by the letter *c*.



A is the point (0, 5)

B is the point (0, –1)

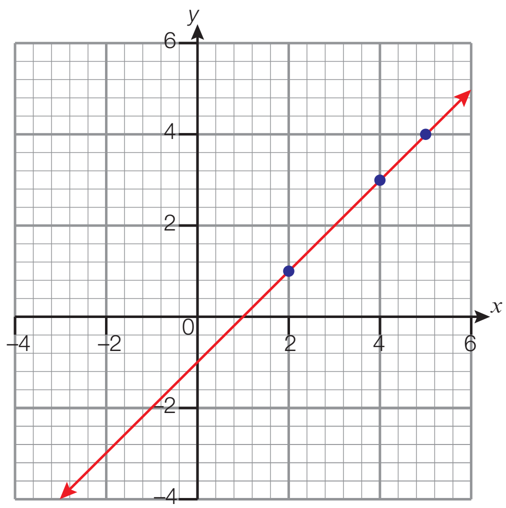
C is the point (0, –3)

If you are asked to give the coordinates of the *y*-intercept, the *x*-coordinate will always be 0.

14.3 Finding the equation of a straight line

The equation of a straight line describes the condition (or rule) that links the *x*- and *y*-coordinates to create that line. For some lines it is simple to look at the diagram or the coordinates and note the link.

If the points (5, 4), (4, 3) and (2, 1) are plotted on a graph, they form a line. What is the rule that connects the *x*- and *y*-coordinates?



(x, y)	x	y in terms of x
(2, 1)	2	$2 - 1 = 1$
(4, 3)	4	$4 - 1 = 3$
(5, 4)	5	$5 - 1 = 4$

The *y*-coordinate is always one less than the *x*-coordinate.

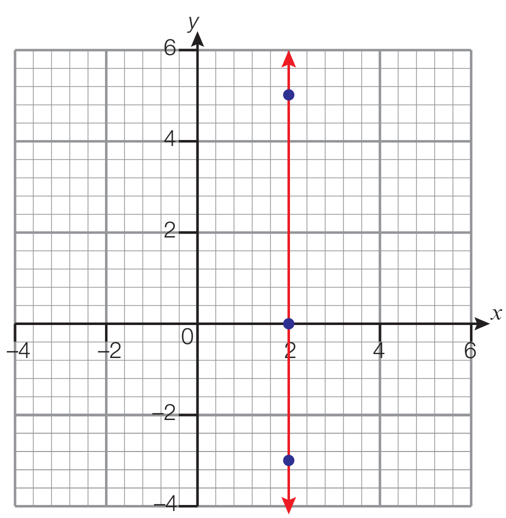
The equation of the line is $y = x - 1$.

Look at the gradient and *y*-intercept on the graph.

The gradient = 1.

The *y*-intercept is (0, −1).

The points (2, 0), (2, 5) and (2, −3) are plotted on a graph. Find the rule that connects them.



x	y
2	−3
2	0
2	5

The *x*-coordinate is always 2; only the *y*-coordinate changes.

The equation of the line is $x = 2$.

The graph shows that the line is vertical.

The gradient of the line is ∞.

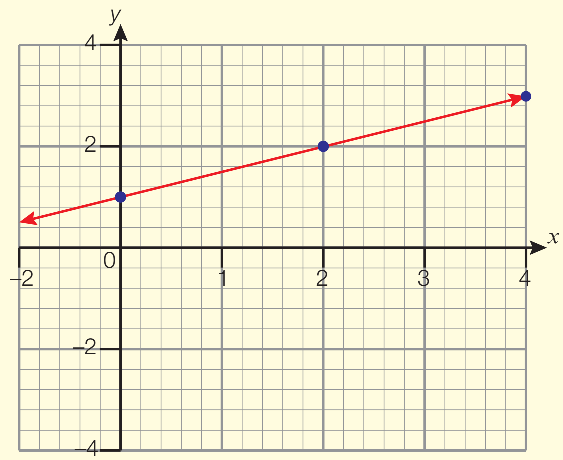
The graph does not cross the *y*-axis, so there is no *y*-intercept.

Worked example 14.3

Q. Plot the points (4, 3), (2, 2) and (0, 1) on a graph and look for the rule that connects the x- and y-coordinates.

(a) Calculate the gradient of the line and the y-intercept.

(b) What connection do the gradient and the y-intercept have with the rule/equation?



A.

x	y	(x, y)	y in terms of x
0	1	(0, 1)	$\frac{1}{2} \times 0 + 1 = 1$
2	2	(2, 2)	$\frac{1}{2} \times 2 + 1 = 1 + 1 = 2$
4	3	(4, 3)	$\frac{1}{2} \times 4 + 1 = 2 + 1 = 3$

Check this rule for $x = 3$:
$y = \frac{1}{2} \times 3 + 1 = \frac{5}{2}$
You can confirm this value by looking at the graph.

The rule that connects the x- and y-coordinates is $y = \frac{1}{2}x + 1$.

Get the information from the graph.

(a) The gradient of the line is $\frac{1}{2}$.

The y-intercept of the line is (0, 1).

(b) The gradient has the same value as the number (**coefficient**) multiplying the x.

The intercept has the same value as the **constant**.

Exercise 14.2

1. Plot the following sets of points on a graph. Determine the rule connecting the x- and y-coordinates of the points and hence find the equation of the line passing through the points.

   (a) $(-1, -2), (0, 0), (2, 4)$

   (b) $(-1, -1), (0, 1), (1, 3)$

   (c) $(1, 2), (2, 5), (3, 8)$

   (d) $(2, 3), (3, 7), (4, 11)$

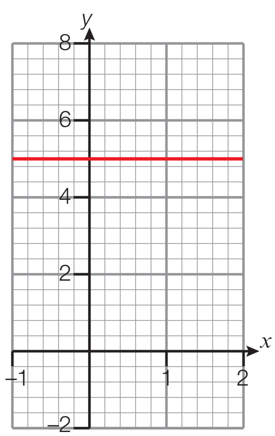
   (e) $(1, -3), (2, -6), (3, -9)$

   (f) $(1, -2), (2, -5), (3, -8)$

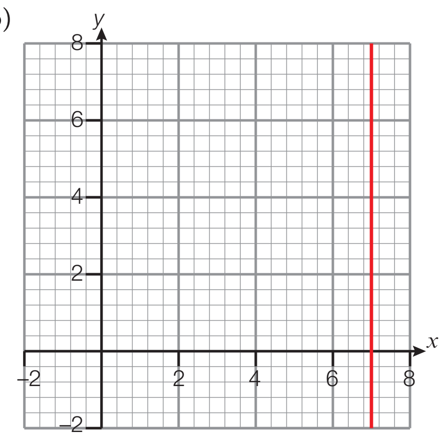
   (g) $(-2, 6), (0, 4), (4, 0)$

2. By selecting two or three suitable points on the line, determine the equation of each of the following straight lines in the graphs (a) to (e). The lines are coloured red.

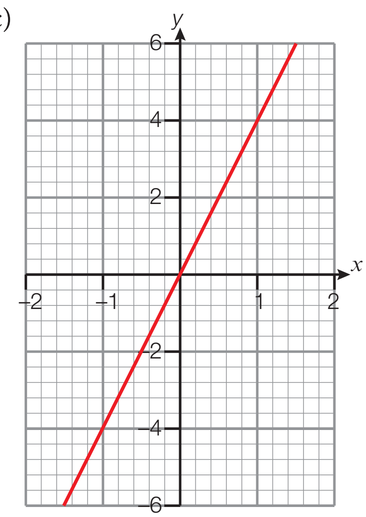
(a)



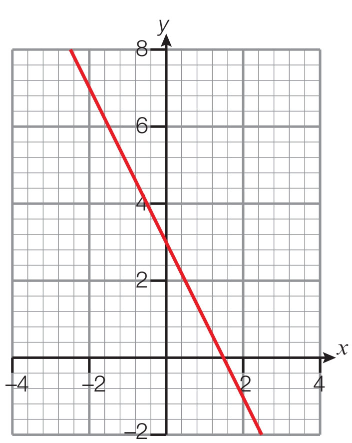
(b)



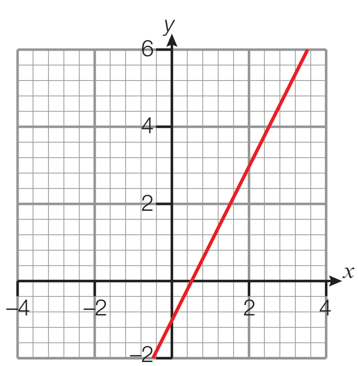
(c)



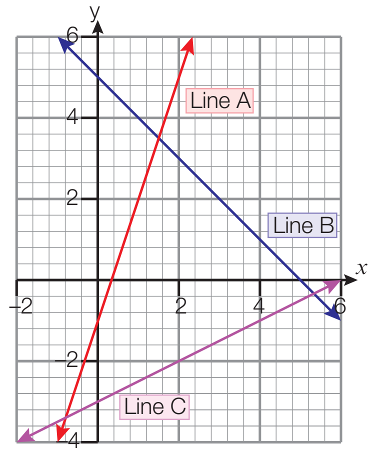
(d)



(e)



3. Find the equations of line A, line B and line C.



14.4 The equation of a straight line

The general equation of a straight line can be given in different forms. In this course you should be able to recognise:

$y = mx + c$	the 'gradient-intercept' form
$ax + by + d = 0$	the 'general' form.

The 'gradient-intercept' form: $y = mx + c$

If the equation is in this form 'm' represents the gradient and '$+ c$' represents the y-intercept.

hint

If the equation is in $y = mx - c$ then '$-c$' represents the y-intercept.

Depending on the information that you are given, you can find the equation of a particular line, as shown in the examples below.

Worked example 14.4

Q. The gradient of a line is $\frac{4}{3}$ and the line crosses the y-axis at (0, –2). Find the equation of the line.

A. If the gradient is $\frac{4}{3}$ then $m = \frac{4}{3}$.

If the line crosses the y-axis at (0, –2) then the y-intercept is –2.

So, the equation of the line is $y = \frac{4}{3}x - 2$.

In this case, the intercept has not been given and must be calculated by substituting known values into $y = mx + c$ and solving the equation for c.

Worked example 14.5

Q. The gradient of a line is –3, and the line passes through the point (2, 5). Find the equation of the line.

Replace y by the value of the y-coordinate of the point, x by the value of the x-coordinate and m by the value of the gradient.

A. $y = mx + c$

$5 = -3 \times 2 + c$

$5 = -6 + c$

$11 = c$

The equation of the line is $y = -3x + 11$.

Worked example 14.6

Q. A line passes through the points A(–3, 4) and B(1, –6). Find the equation of the line AB.

The line is passing through two points but the gradient and the y-intercept have not been given, so we need to calculate them.

A. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{(-6-4)}{(1-(-3))}$

$= \dfrac{-10}{4} = \dfrac{-5}{2}$

Once the gradient has been calculated, you can substitute known values into $y = mx + c$ to solve for c. Choose either A or B as the point. The result will be the same whichever point you use.

Using point A:

$4 = \dfrac{-5}{2} \times -3 + c$

$4 = \dfrac{15}{2} + c$

$4 - \dfrac{15}{2} = c$

$\dfrac{-7}{2} = c$

The equation of the line is $y = -\dfrac{5}{2}x - \dfrac{7}{2}$.

The 'general' form: $ax + by + d = 0$

The general form of a linear equation is $ax + by + d = 0$, where a, b and d are integers.

The general form can be useful because you can use it to write an equation without rational numbers (fractions). If you are given the gradient-intercept form of an equation and it contains fractions, you can rearrange it into the general form to remove the fractions and make it easier to work with. You can also rearrange the general form into the gradient-intercept form.

RR *You were reminded how to rearrange equations in Learning links 2A in Chapter 2.*

Worked example 14.7

Q. Rearrange the following equations into the form $ax + by + d = 0$.

(a) $y = -2x - 1$ (b) $\frac{1}{2}y = 4x - 3$ (c) $\frac{y}{3} = -\frac{x}{2} + 2$

A. (a) $y = -2x - 1$

$y + 2x = -1$

Add $2x$ to both sides of the equation.

$y + 2x + 1 = 0$

Add 1 to both sides of the equation.

(b) $\frac{1}{2}y = 4x - 3$

$y = 8x - 6$

Multiply each term by 2.

$y - 8x + 6 = 0$

Subtract $8x$ and add 6 to both sides of the equation.

(c) $\frac{y}{3} = -\frac{x}{2} + 2$

$2y = -3x + 12$

Multiply each term by 6, the common denominator of $\frac{1}{2}$ and $\frac{1}{3}$.

$3x + 2y - 12 = 0$

Add $3x$ and subtract 12 from both sides of the equation.

Worked example 14.8

Q. A straight line passes through the points A(3, 4) and B(5, 10).

(a) Find the gradient of the line segment AB.

(b) Find the equation of the line AB. Give your answer in the form $y = mx + c$.

continued . . .

(c) Find the equation of the line parallel to AB that passes through the point C(–2, 1).

(d) Find the equation of the line perpendicular to AB that passes through the point D(0, –1). Give your answer in the form $ax + by + d = 0$.

> Calculate the gradient using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

A. (a) $m_{AB} = \dfrac{4-10}{3-5} = \dfrac{-6}{-2} = 3$

> Use one of the points to substitute values into the gradient-intercept equation; solve the equation for c.

(b) Using A(3, 4)

$$y = mx + c$$
$$4 = 3 \times 3 + c$$
$$4 - 9 = c$$
$$-5 = c$$

Equation of AB is
$y = 3x - 5$

> If the lines are parallel, you know that $m = 3$ and you can substitute this value and the coordinates of point C into $y = mx + c$ to solve for c.

(c) If lines are parallel, then $m = 3$ for both lines

Using C(–2, 1)

$$1 = 3 \times -2 + c$$
$$1 = -6 + c$$
$$7 = c$$

Equation of line through C is
$y = 3x + 7$

> $m_1 \times m_2 = -1$
> $3 \times -\dfrac{1}{3} = -1$

(d) If $m = 3$, the gradient of the perpendicular line $= -\dfrac{1}{3}$

> Substitute the coordinates of point D into $y = mx + c$ to solve for c.

Using D (0, –1)

$$-1 = -\frac{1}{3} \times 0 + c$$
$$-1 = c$$

Equation of line through D is

> Rearrange $y = mx + c$ into the general form $ax + by + d = 0$. Multiply each term by 3 to remove the fraction.

$$y = -\frac{1}{3}x - 1$$
$$3y = -x - 3$$
$$x + 3y + 3 = 0$$

> Descartes demonstrated that geometric problems could be described in terms of algebra, and algebra problems in terms of geometry. Does this tell you that some problems can be seen from two entirely different viewpoints?

Exercise 14.3

1. Find the equation of the straight line through the given point with the given gradient.

   (a) (3, 7); 6

   (b) (0, 8); 4

   (c) (–2, 4); –5

(d) $(1, 0); -3$

(e) $\left(\frac{3}{4}, -3\right); 2$

(f) $\left(-5, \frac{3}{4}\right); \frac{5}{7}$

2. Find the equation of the straight line passing through the following pairs of points.

(a) $(0, 4)$ and $(-2, 8)$

(b) $(-3, 5)$ and $(7, 3)$

(c) $(2, -1)$ and $(9, 0)$

(d) $\left(7, \frac{1}{3}\right)$ and $\left(11, \frac{5}{6}\right)$

(e) $(-6, -3)$ and $(-1, 4)$

(f) $\left(\frac{1}{2}, \frac{3}{4}\right)$ and $(0, -1)$

3. The points A and B have coordinates $(2, 5)$ and $(4, 9)$ respectively. A straight line l_1 passes through A and B.

(a) Find the equation of line l_1 in the form $ax + by + d = 0$.

The line l_2 passes through the point $(3, -5)$ and has gradient $\frac{2}{7}$.

(b) Find the equation of l_2

4. Find the equation of the line that is:

(a) parallel to the line $y = 7x - 1$ and passes through the point $(2, -9)$

(b) passes through the point $(-4, 7)$ and is parallel to the line $5x + 2y = 3$

(c) passes through $\left(3, \frac{1}{5}\right)$ and is parallel to the line $11 - 3x - 2y = 0$

(d) parallel to the line $5x = 9 - 8y$ and passes through the origin.

5. Find the equation of the line passing through the point:

(a) $(6, -1)$ and perpendicular to the line $y = 3x$

(b) $(11, 15)$ and perpendicular to the line $7x = 2y$

(c) $(0, 9)$ and perpendicular to the line $4x - 3y = 13$

(d) $\left(-\frac{1}{5}, 0\right)$ and perpendicular to the line $3x + 2y - 12 = 0$.

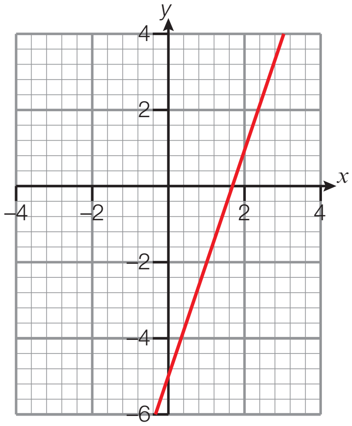
6. (a) Find the gradient of the line joining the points A$(-1, 2)$ and B$(3, 4)$.

(b) Hence find the equation of the line AB in the form $y = mx + c$.

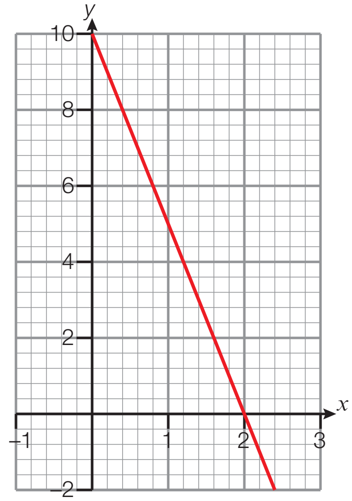
(c) Find the equation of the line CD, perpendicular to AB and passing through point C$(5, 9)$.

7. Find the equation of each of the following lines in the form $y = mx + c$.

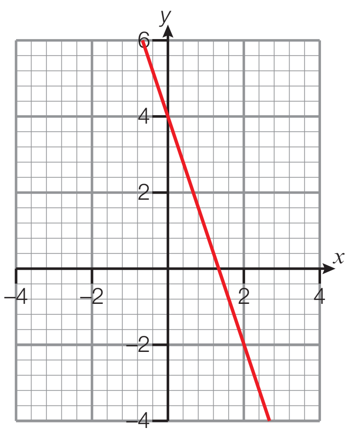
(a)



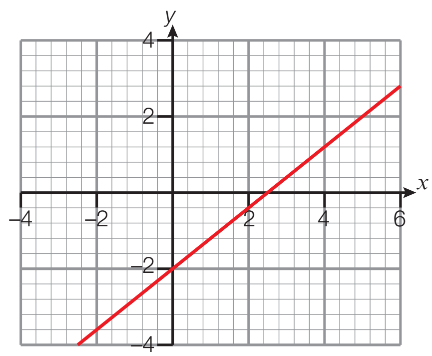
(b)



(c)



(d)



8. (a) Find the gradient of the line joining the points A(1, –2) and B(3, –4).

(b) Hence find the equation of the line joining the points A and B in the form $y = mx + c$.

(c) Rearrange the equation of the line to the form $ax + by + d = 0$, where a, b and d are integers.

14.5 Drawing a straight line graph from an equation

The equation of a line gives the unique rule that allows you to draw that line, and that line only.

Using $y = mx + c$

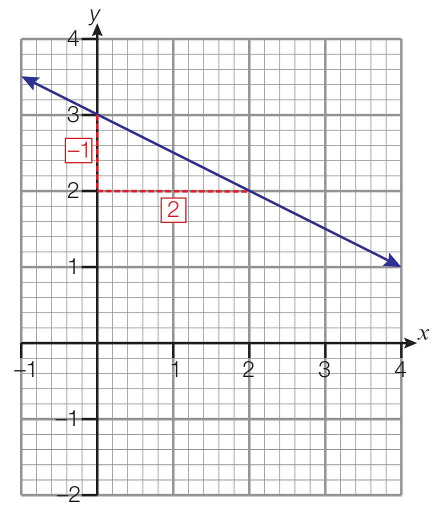
There are three methods for drawing a straight line graph using $y = mx + c$. You can:

1. Use the values of m and c.
2. Calculate a table of values using the equation and plot them.
3. Use your GDC.

To draw the line with equation $y = -\frac{1}{2}x + 3$

Method 1:

$m = -\frac{1}{2}$, $c = 3$, so the line passes through the point (0, 3) with a gradient of $-\frac{1}{2}$.

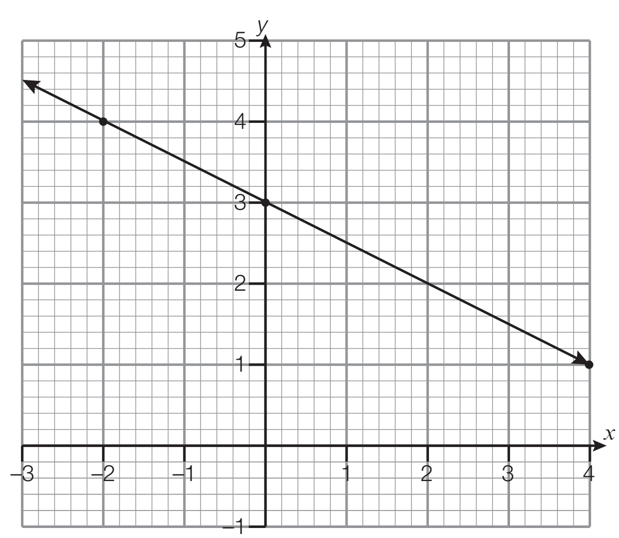


Method 2:

Draw up a table of values and then plot the points. It is helpful to calculate at least three points, as this will confirm that you have a straight line.

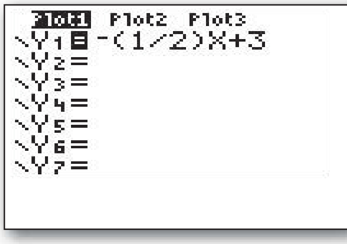
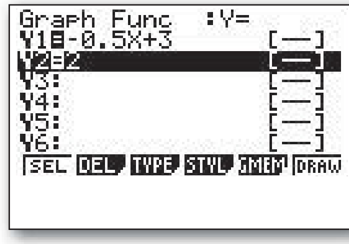
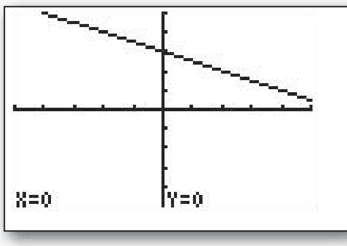
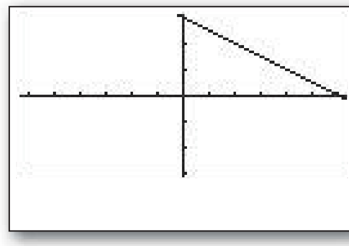
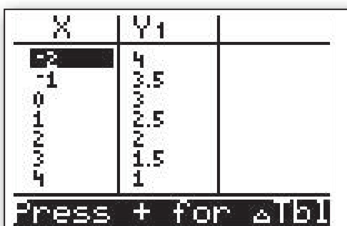
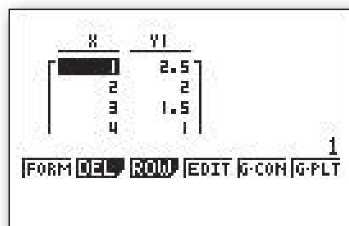
x	–2	0	4
$-\frac{1}{2}x + 3 = y$	$-\frac{1}{2}(-2) + 3 = 4$	$-\frac{1}{2} \times 0 + 3 = 3$	$-\frac{1}{2} \times 4 + 3 = 1$

Plot the points (–2, 4), (0, 3) and (4, 1).



Method 3:

Using your GDC (see '*22.2.G Graphs*' on page 645 and '*14.1 Accessing the table of coordinates from a plotted graph*' on page 678 of the GDC chapter for a reminder if you need to):

	TEXAS	CASIO
i. Enter the equation.		
ii. Draw the line.		
iii. Use the table to check the coordinates of your points.		

Using $ax + by + d = 0$

There are three methods for drawing a straight line graph using $ax + by + d = 0$. You can:

1. Use the equation to find the intercepts on the x- and y-axes.
2. Rearrange the general equation into the gradient-intercept form and use the values of m and c.
3. Rearrange the general equation into the gradient-intercept form and use your GDC.

To draw the line with equation $3x - 4y - 12 = 0$

Method 1:

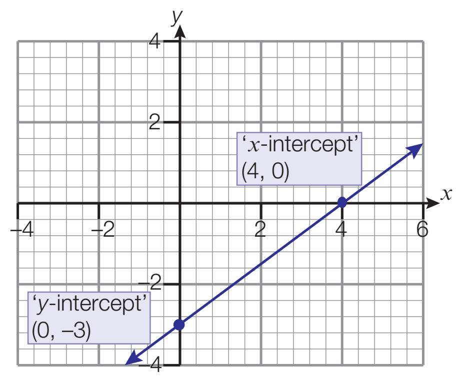
Find the intercepts on the x- and y-axes by calculating the points where $x = 0$ and $y = 0$ and drawing a line through those points.

When $x = 0$, $3x = 0$ so $3x - 4y - 12 = 0 \Rightarrow$

$$-4y - 12 = 0$$
$$-4y = 12$$
$$y = -3 \quad \text{Plot } (0, -3)$$

When $y = 0$, $4y = 0$ so $3x - 4y - 12 = 0 \Rightarrow$

$$3x - 12 = 0$$
$$3x = 12$$
$$x = 4 \quad \text{Plot } (4, 0)$$



Method 2:

Rearrange into the gradient-intercept form with y as the subject of the equation, then use the values of m and c to draw the line.

$$3x - 4y - 12 = 0$$

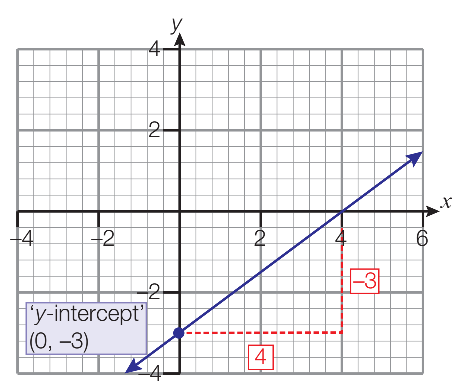
Add $4y$ to both sides of the equation.

$$3x - 12 = 4y$$

Divide each term by 4.

$$\tfrac{3}{4}x - 3 = y$$

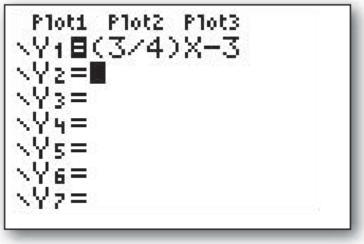
If $y = \frac{3}{4}x - 3$, draw the line with an intercept of $(0, -3)$ and gradient of $\frac{3}{4}$.

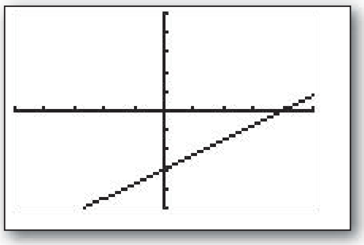


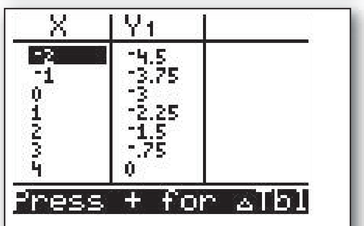
Method 3:

When using your GDC, you must use the equation in the form $y = mx + c$ so, from method 2 above, draw $y = \frac{3}{4}x - 3$ on your GDC and transfer the line onto your graph paper, using the table function.

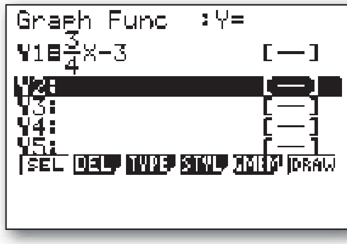
TEXAS

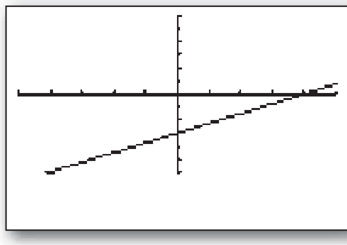


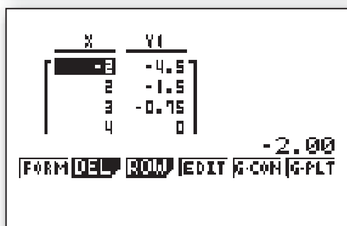




CASIO







Worked example 14.9

Q. Draw a Cartesian plane with axes $-6 \le x \le 6$ and $-8 \le y \le 6$.

(a) On the Cartesian plane, draw the line given by the equation $y - 2x + 7 = 0$.

(b) Rearrange the equation of the line into the gradient-intercept form.

(c) Write down the gradient of the line.

(d) Find the equation of a line parallel to the original, which passes through the point $(-2, 1)$. Draw this line.

(e) Find the equation of a line perpendicular to the original that passes through the point $(-1, 4)$. Draw this line.

To draw the line with equation $y - 2x + 7 = 0$, find when the line crosses each axis.

When $x = 0$, $y = -7$.

When $y = 0$, $x = 3.5$.

The line is drawn through $(0, -7)$ and $(3.5, 0)$.

The gradient-intercept form is $y = mx + c$. Add $2x$ to both sides and subtract 7 from both sides.

In the form $y = mx + c$, m is the gradient.

If the new line is parallel then its gradient is also +2. Substitute $x = -2$ and $y = 1$ into $y = mx + c$ and solve for c.

If the new line is perpendicular, then the gradients multiplied together should equal −1. Create an equation to solve.

Substituting $x = -1$ and $y = 4$ into $y = mx + c$ and solve for c.

continued . . .

A. (a) See (a) in dark blue in the graph below for line $y - 2x + 7 = 0$.

(b) $y - 2x + 7 = 0$

$\Rightarrow y = 2x - 7$

(c) The gradient of line is +2.

(d) $1 = 2 \times (-2) + c$

$1 = -4 + c$

$c = 5$

The new equation is $y = 2x + 5$

See line (d) in pink on the graph below.

(e) $m_2 \times 2 = -1$

$m_2 = -\frac{1}{2}$

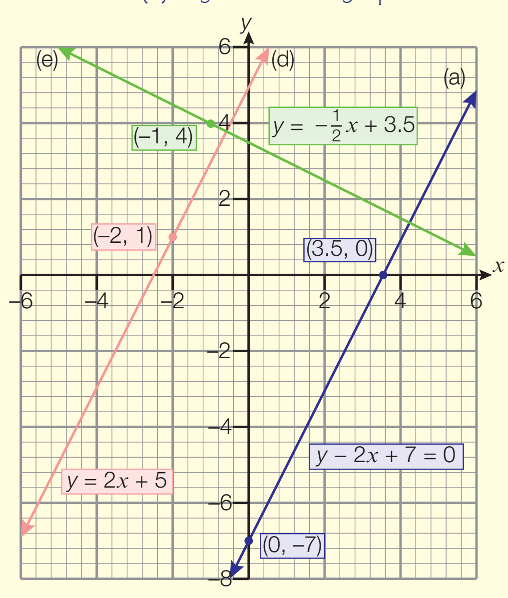
$4 = -\frac{1}{2} \times -1 + c$

$4 = \frac{1}{2} + c$

$c = 3.5$

The new equation is $y = -\frac{1}{2}x + 3.5$

See line (e) in green on the graph below.



Exercise 14.4

1. Draw the lines of the equations given below.

   (a) $y = 2x - 3$ (b) $y = 1.5x + 4$

   (c) $y = 6 - x$ (d) $y = 10 - 7x$

   (e) $2y = 8 + 5x$ (f) $5y = 3x - 9$

2. Draw the following lines:

   (a) $4y = 3x - 12$ (b) $x + y = 7$

   (c) $2x + 3y = 6$ (d) $6y - 7 = 5x$

   (e) $3x + 4y - 2 = 0$ (f) $4x - y - 3 = 0$

   (g) $7x + 3y - 4 = 0$

3. (a) Draw the line with the equation $y = 3x + 4$.

   (b) Find the equation of the line parallel to $y = 3x + 4$ that passes through the point (2, 7).

4. (a) Draw the line with the equation $y = 4 - 5x$.

   (b) Find the equation of the line parallel to $y = 4 - 5x$ and passing through the point (−1, 2).

   (c) Find the equation of the line passing through the point (3, 8) which is perpendicular to the line $y = 4 - 5x$.

5. Solve the following simultaneous equations. In each case use your GDC to draw the pair of straight lines and write down the point of **intersection**.

   (a) $y = 7x - 4$ and $y = 4$ (b) $y = 4x - 1$ and $y = -2x + 2$

   (c) $y = 6 - 5x$ and $y = 4x - 12$ (d) $3y + x = 1$ and $5y = x + 7$

   (e) $x - 2y = 2$ and $3x - 4y = 8$ (f) $2x + 3y - 7 = 0$ and $15x - 7y - 9 = 0$

RR ***Chapter 2 gives methods for solving pairs of linear equations. One of the methods described uses a graph, and the solution is found where the two lines intersect.***

Summary

You should know:

- how to calculate the gradient and y-intercept of a line
- that the equation of a line in two dimensions can be given in two forms:

  – the gradient-intercept form, $y = mx + c$

  – the general form, $ax + by + d = 0$
- that parallel lines have the same gradient ($m_1 = m_2$) and two lines are perpendicular if the product of their gradients is -1 ($m_1 \times m_2 = -1$)
- how to use information from a graph to create an equation of the line
- how to draw a graph using the equation of the line.

Mixed examination practice

Exam-style questions

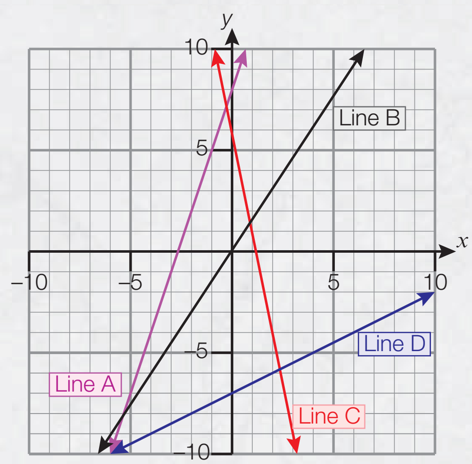
1. Find the equation of the straight lines through the point (–3, 7) with the following gradients:

(a) 2

(b) –5

(c) $-\frac{3}{4}$ (Leave your answer to part (c) in the form $ax + by + c = 0$.)

2. The graph below shows four lines; A, B, C and D. Write down the equation of each of the lines in the form $y = mx + c$.



3. The points C and D have coordinates (–1, –8) and (1, 2) respectively.

(a) Find the gradient of the line joining the points C and D.

(b) Hence find the equation of the line joining the points C and D in the form $y = mx + c$.

4. The line L_1 has the equation $2y + 3x = 6$.

(a) Draw the graph of line L_1.

(b) Find the equation of the line L_2 which passes through the point (–3, 0) and is perpendicular to the line L_1.

(c) Determine the equation of line L_3, passing through the point (4, –7) and parallel to L_1.

5. The line L_1 has the equation $4x + 5y - 20 = 0$.

(a) Draw the graph of line L_1.

(b) Write down the equation of any line which is:

(i) parallel to L_1

(ii) perpendicular to L_1.

A second line, L_2, has equation $20y - 15x - 18 = 0$.

(c) Draw the graph of line L_2.

(d) Find the point of intersection of the two lines.

(e) Hence state the solutions of the following pair of simultaneous equations:

$4x + 5y - 20 = 0$ and $20y - 15x - 18 = 0$.

Past paper questions

1. (a) Write down the gradient of the line $y = 3x + 4$. *[1 mark]*

(b) Find the gradient of the line which is perpendicular to the line $y = 3x + 4$. *[1 mark]*

(c) Find the equation of the line which is perpendicular to $y = 3x + 4$ and which passes through the point (6, 7). *[2 marks]*

(d) Find the coordinates of the point of intersection of these two lines. *[2 marks]*

[Total 6 marks]

[May 2008, Paper 1, TZ2, Question 6] (© *IB Organization 2008*)

2. Three points are given A(0, 4), B(6, 0) and C(8, 3).

(a) Calculate the gradient (slope) of line AB. *[2 marks]*

(b) Find the coordinates of the **midpoint**, M, of the line AC. *[2 marks]*

(c) Calculate the length of line AC. *[2 marks]*

(d) Find the equation of the line BM giving your answer in the form $ax + by + d = 0$ where a, b and $d \in \mathbb{Z}$. *[5 marks]*

(e) State whether the line AB is perpendicular to the line BC showing clearly your working and reasoning. *[3 marks]*

[Total 14 marks]

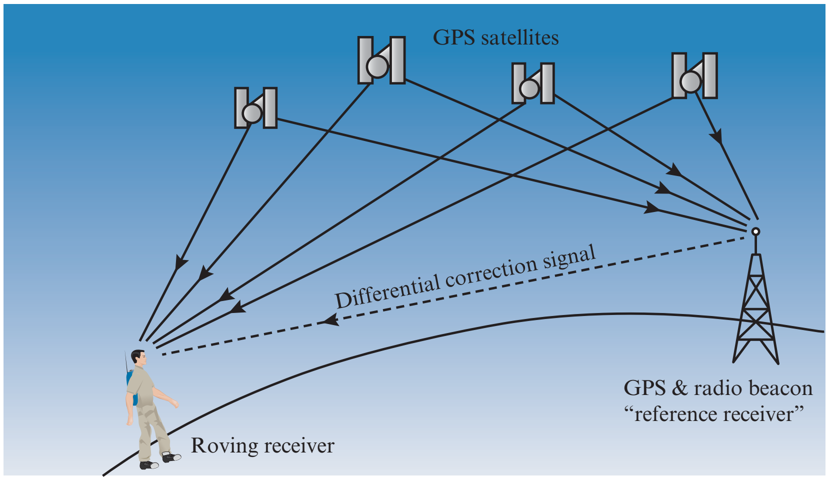
[Nov 2006, Paper 2, TZ0, Question 2] (© *IB Organization 2006*)

hint

In question 2, part (b), the midpoint M is halfway along the line AC.

Chapter 15 Trigonometry

Trigonometry is a practical application of mathematics that is used in fields as diverse as navigation, engineering, map-making and building.



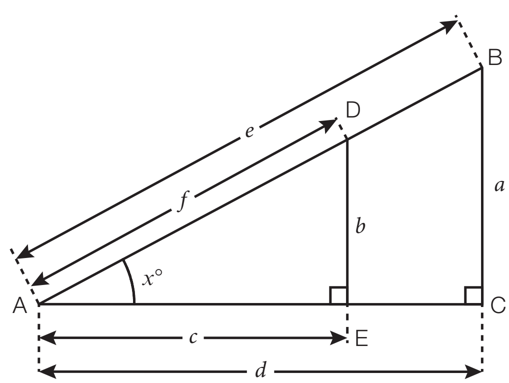
In the most basic terms, a receiver of a global positioning system (GPS) dermines where it is on Earth by calculating the distance between at least three satellites in space and the receiver on the ground (artwork adapted from http://www.thegeeksclub.com/how-gps-works/).

In this chapter you will learn:

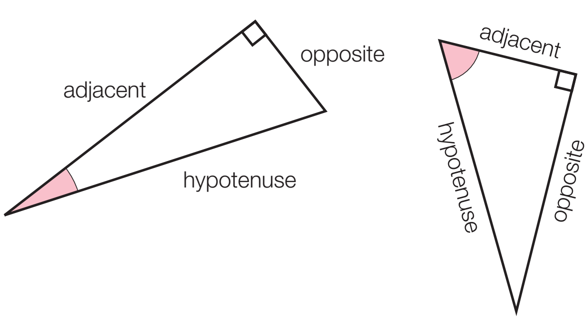
- how to use the sine, cosine and tangent ratios to find unknown sides and angles of right-angled triangles
- about angles of elevation and depression
- how to use the sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- how to use the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$ and $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
- about an alternative formula for finding the area of a triangle, $A = \frac{1}{2}ab\sin C$
- the best way of sketching labelled diagrams from verbal statements.

Trigonometry uses the fact that, as long as the shape of a triangle does not change, the ratios of its sides remain the same.

In this diagram, triangles ADE and ABC are similar; the angle $x°$ is the same for both. So, $\frac{a}{b} = \frac{d}{c} = \frac{e}{f}$.



In any **right-angled triangle**, the sides are given the same names:



The **hypotenuse** is always opposite the right angle (no matter how the triangle is oriented). It is the longest side.

The opposite side is opposite the given angle.

The adjacent side is beside the given angle.

15.1 Trigonometric ratios

exam tip

Always make sure that your GDC is set in degrees. If you reset your calculator, it will change the degree measurement to radians. Radians are not used in this course.

There are three standard ratios that have values programmed into your GDC. Their full names are **sine**, **cosine** and **tangent**, but these are frequently shortened to sin, cos and tan, respectively. The short versions are printed on your calculator keys.

The ratios are defined as:

$$\text{sin(e) } \theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{cos(ine) } \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{tan(gent) } \theta = \frac{\text{opposite}}{\text{adjacent}}$$

exam tip

Make sure that you learn the trigonometric ratios – they are not given in the Formula booklet. SOH, CAH, TOA is a useful way to remember them; or you could make up your own reminder.

When you are using a **trigonometric ratio**, the most important step is choosing the correct one! Once you have identified the correct ratio to use:

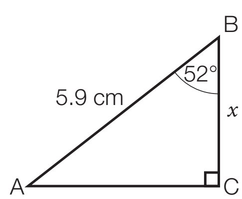
- write down the formula for that ratio
- insert the values that you know
- complete your calculation.

If you are solving a problem that involves right-angled triangles, you can use trigonometry to find:

- the length of **a side**, given that you know the length of one other side and the size of an angle
- the size of **an angle,** given that you know the lengths of two sides.

The uses of trigonometry are wide-ranging and unexpected. Their use has been extended by the deeper understanding of sine and cosine curves. The concept of the right-angled triangle has been used in map-making and land surveying for centuries. The techniques of trigonometry now support such varied uses as: aviation, architecture, satellite systems and GPS, oceanography, digital imaging and the way that music is downloaded from the internet.

For example, to find side x in the triangle ABC below first check the information:



o(pposite) no ✗

a(djacent) yes (x) ✓

h(ypotenuse) yes (5.9cm) ✓

Use CAH = cosine.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 52° = \frac{x}{5.9}$$

$$x = 5.9 \times \cos 52°$$

$$x = 3.63\text{ cm}$$

If you know the lengths of two sides and need to know the length of the third side, use **Pythagoras' theorem**.

GDC

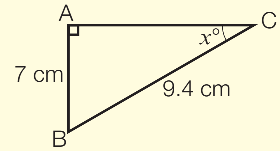
If you need a reminder of how to use the trigonometric keys on your GDC, see section '*22.2B The second and third functions of a calculator key*' on page 642 of the GDC chapter.

Worked example 15.1

Q. For the given triangle, calculate:

(a) the size of angle x

(b) the length of the adjacent side.



o – yes; a – no; h – yes; so use sin ratio. Substitute in known values and solve for x.

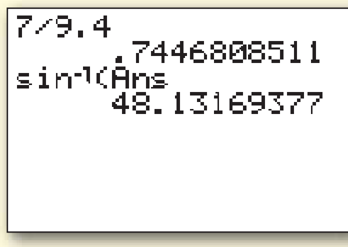
A. (a) $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$

$\sin x = \frac{7}{9.4} = 0.7447$

$x = 48.1°$

To get the angle in degrees, you need to use inverse sin; this is the $\sin^{-1}$ key on your GDC.

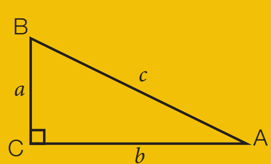
TEXAS



CASIO

Pythagoras' theorem states that 'the square on the hypotenuse is equal to the sum of the squares on the other two sides.'

$c^2 = a^2 + b^2$



(b) $AC^2 = 9.4^2 - 7^2$

$AC^2 = 39.36$

$AC = \sqrt{39.36}$

$AC = 6.27\,\text{cm}$

You know the lengths of two sides.

Use Pythagoras' theorem to find the length of the third side.

Worked example 15.2

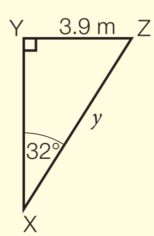
Q. In the triangle XYZ, $Y\hat{X}Z = 32°$ and $YZ = 3.9\,\text{cm}$.

(a) Find the length of XZ, the hypotenuse.

(b) Calculate the length of XY.

First, sketch a diagram and label it with the information from the question. Then identify the ratio needed and solve the equation.

A.



To find XZ, you are using the opposite side and the hypotenuse. Choose SOH.

continued . . .

(a) $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\sin 32° = \dfrac{3.9}{y}$

$y = \dfrac{3.9}{\sin 32°}$

$y = 7.36\text{ m}$

Using Pythagoras' theorem.

(b) $XY^2 = XZ^2 - YZ^2$

$= 7.36^2 - 3.9^2$

$XY = 6.24\text{ m}$

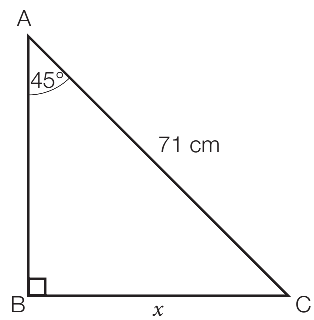
GDC

Remember to use the square root ($\sqrt{\ }$) key.

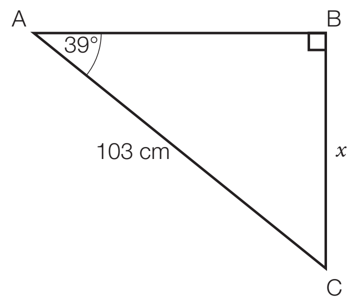
Exercise 15.1

1. For each of the following triangles work out the length of the side labelled x.

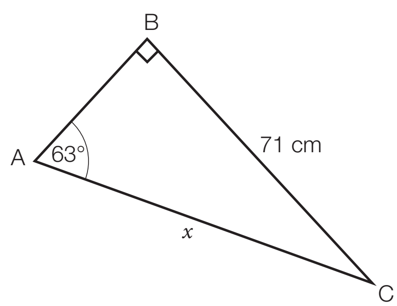
(a)



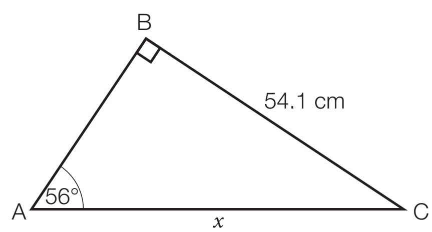
(b)



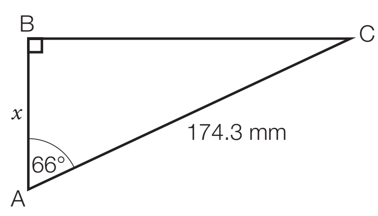
(c)



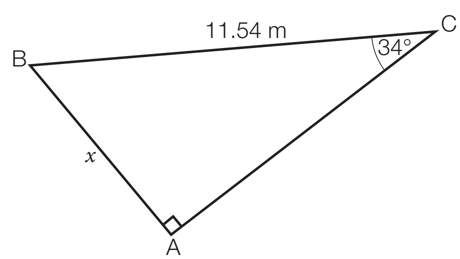
(d)



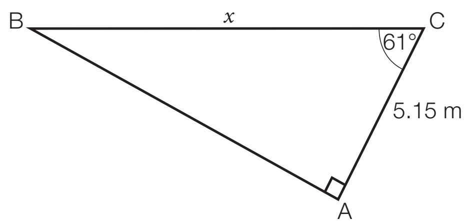
(e)



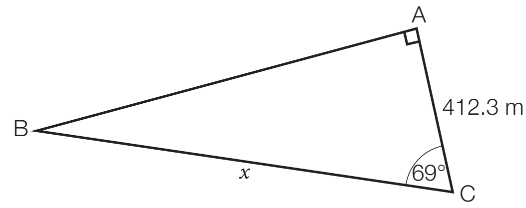
(f)



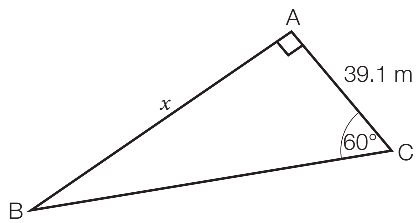
(g)



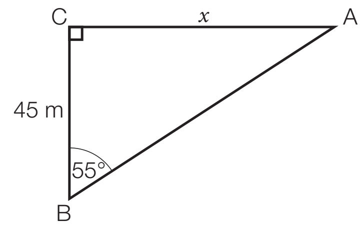
(h)



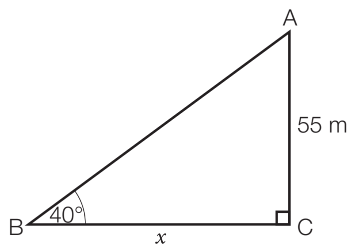
(i)



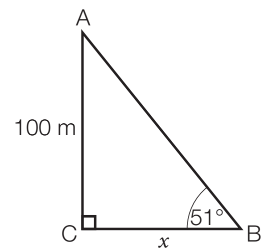
(j)



(k)

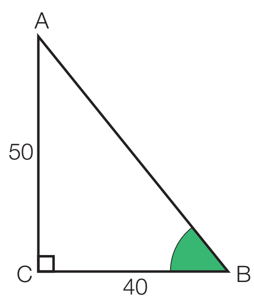


(l)

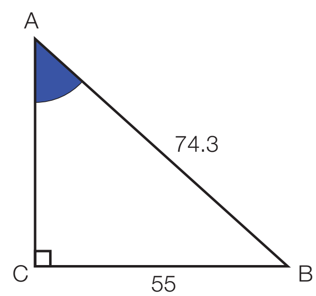


2. Calculate the size of the shaded angle in each of the triangles below.

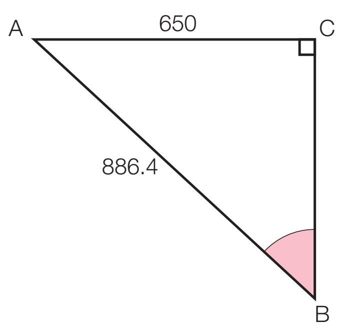
(a)



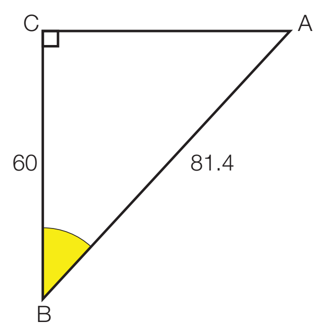
(b)



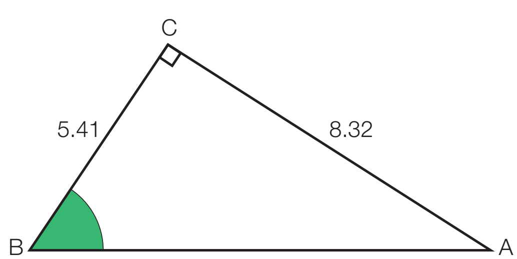
(c)



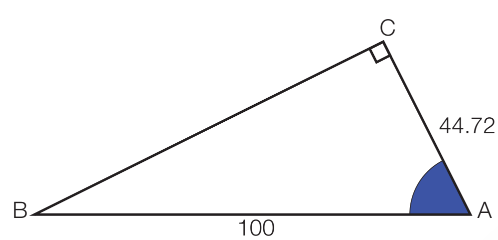
(d)



(e)



(f)



exam tip

Always sketch a clear diagram for problem-solving questions involving trigonometry.

3. Triangle PQR is isosceles with PQ = PR = 35 cm. Angle $Q\hat{P}R = 84^{\circ}$. Find the length of the perpendicular from P to QR.

4. A ladder 6 metres long leans against a vertical wall. The ladder makes an angle of 54° with the wall. How far up the wall does the ladder reach?

5. A ladder 8 metres long is placed against a vertical wall. If the ladder reaches 6.3 metres up the wall, calculate the angle the ladder makes with the wall.

6. Germaine is flying a kite. She pins the end of the kite string to the ground so that the string is pulled straight as the wind pushes on the kite. If the length of the kite string is 7.8 metres and the kite has a vertical height of 5.9 metres above the ground, what angle does the string of the kite make with the horizontal ground?

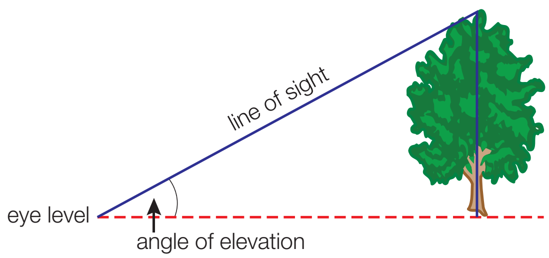
exam tip

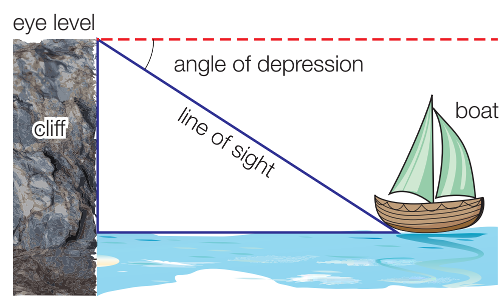
If you draw a diagram for problems that use these terms, it will help you to identify the correct angle.

15.2 Angles of elevation and depression

If you lie on the ground so that the base of a tree is at eye level and look up, the angle that you look through is called the **angle of elevation**.

If you lie on the ground at a cliff top and look down at a boat on the water, the angle that you look through is called the **angle of depression**.

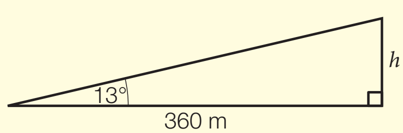




Worked example 15.3

Q. Una is lying down on flat ground, 360 m from the base of a television mast. If the angle of elevation from Una to the top of the mast is 13.0°, how high is the mast?

A.



continued . . .

o – yes

a – yes

h – no

Use TOA.

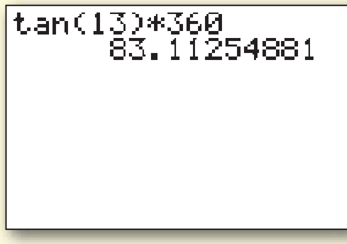
$\tan x = \dfrac{\text{opposite}}{\text{adjacent}}$

$\tan 13° = \dfrac{h}{360}$

$h = \tan 13° \times 360$

$h = 83.1\text{ m}$

TEXAS



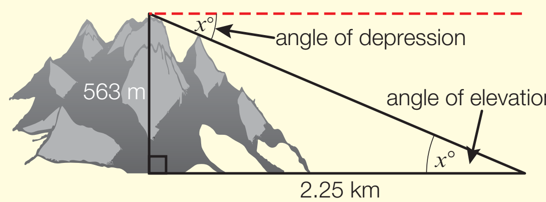
CASIO

Worked example 15.4

Q. Alexis is lying down on top of a hill, looking at a garage in the village below. If the hill is 563 m high, and the village is 2.25 km away, calculate the angle of depression.

The angle of depression between two points = the angle of elevation between those points; from the diagram you can see that the angles are alternate angles, which means they are the same size.

A.



o – yes; a – yes; h – no.
Use TOA.

$\tan x = \dfrac{\text{opposite}}{\text{adjacent}}$

Make sure the units of distance are the same.

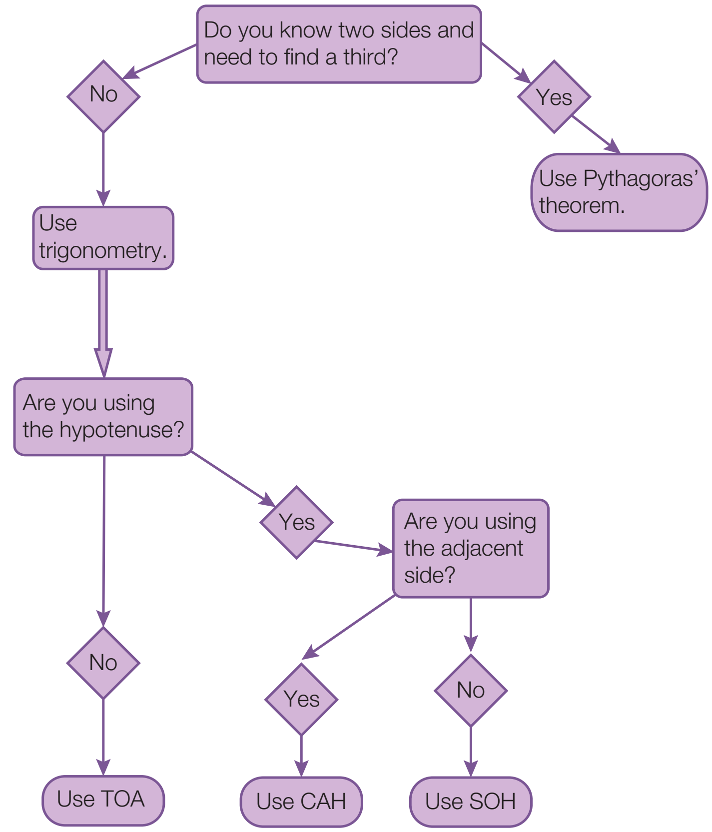
$\tan x = \dfrac{563}{2250}$

$x = 14.0°$

$x = \text{angle of depression} = 14.0°$

Summary

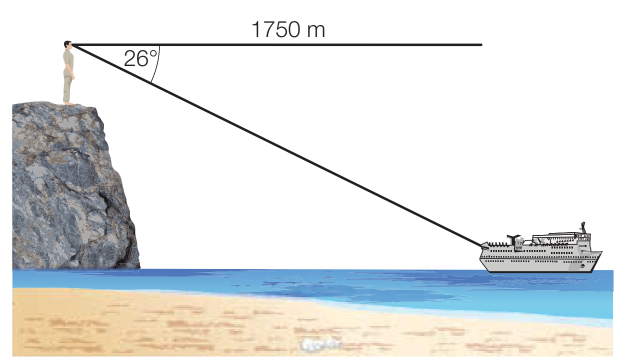
This flow chart can help you to choose the correct trigonometric ratio for any given problem.



Exercise 15.2

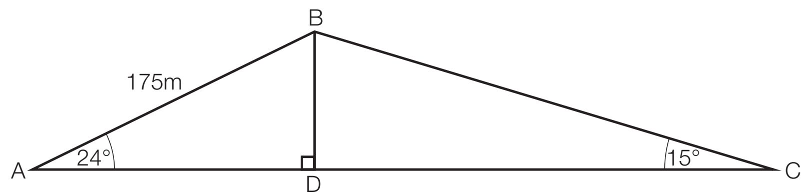
1. From a point on a boat, the angle of elevation to the top of a vertical cliff is 32°. If the horizontal distance of the boat from the bottom of the cliff is 1650 metres, calculate the vertical height of the cliff.

2. A coastguard is standing on a cliff. From the top of the cliff he measures the angle of depression of an approaching fishing trawler to be 26°. The distance between the coastguard and the fishing trawler is 1750 metres.



Given that the coastguard is 1.89 metres tall, find the vertical height of the cliff above the trawler.

3. From a point A, the angle of elevation of the top, B, of a tall building is 24°. From another point C on the opposite side of the building, the angle of elevation of the top of the building is 15°. It is known that the distance between A and B is 175 m.



(a) Calculate the height of the building.

(b) Find the distance between the points A and C.

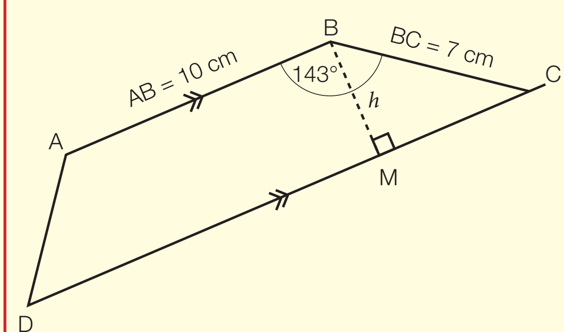
15.3 Harder trigonometry problems

In a practical problem, right-angled triangles can be difficult to identify. It may be necessary to draw in an extra line so that you can use right-angled trigonometry.

Worked example 15.5

Sketch in the line BM; this is the perpendicular height of the trapezium. You can see this creates the right-angled triangle BMC.

Q. Find the height of this trapezium:



You can use the angle relationships formed between parallel lines to find angle $M\hat{B}C$. $A\hat{B}M$ and $B\hat{M}C$ are alternate angles, so are equal. This leaves you with a right-angled triangle, with angle 53° adjacent to the side whose length you want to find. Use cos.

A. $A\hat{B}M = B\hat{M}C = 90°$

$M\hat{B}C = 143 - 90 = 53°$

$\cos 53° = \frac{h}{7}$

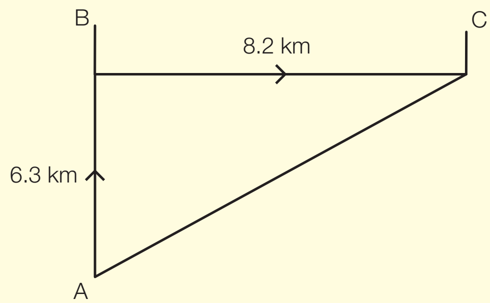
$h = 4.21$ cm

Worked example 15.6

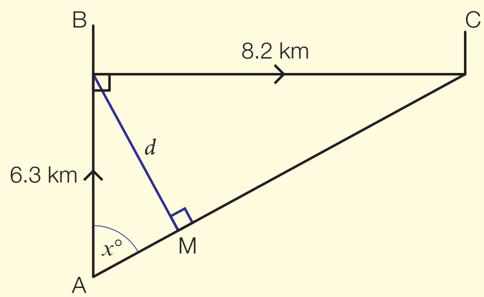
Q. Three friends are hiking. They start at Amville and walk 6.3 km due north to Brent. They plan to turn east and walk another 8.2 km to Cosford. At Brent, one of the friends twists his ankle so the group decides to take the shortest route to the road that runs directly from Amville to Cosford. How far will they have to walk to reach the road?

Sketch a diagram.

A.



Draw the route from B to the road AC and mark the angle that you need to calculate. Label the point where the extra line meets the line AC as M. This creates the right-angled triangle ABM.



Use the triangle ABC to calculate x.

$\tan x = \frac{8.2}{6.3}$

$x = 52.5°$

Use the triangle ABM and the angle x to calculate d.

$\sin 52.5° = \frac{d}{6.3}$

TEXAS

```
tan⁻¹(8.2/6.3)
        52.46519086
sin(Ans)*6.3
        4.995795108
```

CASIO

```
tan⁻¹ (8.2÷6.3)
        52.46519086
sin Ans×6.3
        4.995795108
JUMP DEL ▸MAT MATH
```

$d = 5.00$ km (3 s.f.)

GDC

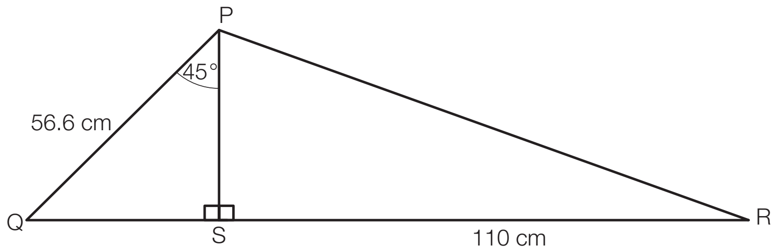
Do not round the answer to the angle calculation too soon. Leave all the figures in your GDC and use them all. (See '*22.2C The Ans/ANS key*' and '*22.2D Using the GDC memory*' on page 643 in the GDC chapter if you need a reminder of how.)

Give the final answer to 3 s.f.

Exercise 15.3

For these questions, if you are not given a diagram, make sure that you draw a clear picture for yourself.

1. In the triangle below, PQ = 56.6 cm, SR = 110 cm and $Q\hat{P}S = 45°$.



Calculate:

(a) the length of PS

(b) the length of QS

(c) the size of angle $Q\hat{P}R$.

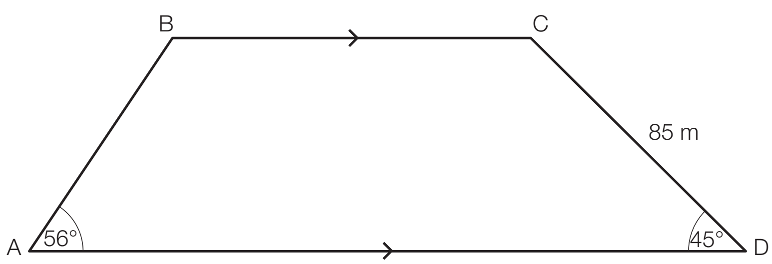
2. Arif, Ben and Chanika are standing in the playground. Arif is standing at point A. Ben is standing 86 m directly north of Arif at point B. Chanika is standing 119 m due east of Arif.

Chanika looks at Arif and then turns through an angle of x degrees to walk towards Ben.

(a) Calculate the distance between Arif and Chanika.

(b) Determine the value of x.

3. ABCD is the plan of a trapezoidal garden. Angles $B\hat{A}D$ and $A\hat{D}C$ are 56° and 45° respectively. Given that the length of CD is 85 m, calculate the length of AB.



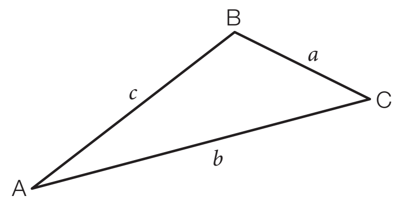
4. Luca and Sasha are playing on a rectangular playing field ABCD. The lengths of AB and BC are 150 m and 80 m respectively. They part, with Luca walking directly along AB and Sasha walking along the diagonal AC. Find the angle between their two routes.

5. David is standing at a point D on the bank of a river. Emma is standing directly opposite David on the other bank at point E. Francis is standing at point F on the same bank as Emma but 70 m downstream along a straight stretch of the river. The angle between DE and DF is 28°. What is the width of the river between David and Emma?

15.4 The sine rule

If you work with triangles that have no right angle, you need new rules and formulae.

To make the formulae easy to use, the triangles are always labelled according to the following convention.



Side 'a' is opposite angle A.

Side 'b' is opposite angle B.

Side 'c' is opposite angle C.

You can use this convention with other letters too, for example:

side 'x' is opposite angle X.

In the standard ABC triangle, the sine rule states that:

$a=\pi r^2$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (use to find the length of a side) or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (use to find the size of an angle)

The sine rule has three ratios but you only need two for each calculation. To make sure that you have the correct pairs, look for one ratio that has two known values and one ratio containing the side or angle that you are looking for.

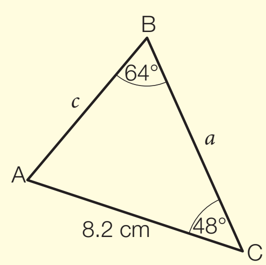
In the example below, there is one known and one unknown quantity in each ratio, so you **cannot** use the sine rule.

$$\frac{129}{\sin A} = \frac{150}{\sin B} = \frac{c}{\sin 28°}$$

If you fill in the known values and cannot find a complete ratio that you can use for the calculation, use the cosine rule (see section 15.5).

Worked example 15.7

Q. Find the length of side AB.



Substitute known values into all the ratios before you decide which pair you need. You need to find the length c, so use the ratio containing c and the ratio with no unknowns.

Cross multiply to solve for c.

continued . . .

A. $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$\frac{a}{\sin A}=\frac{8.2}{\sin 64°}=\frac{c}{\sin 48°}$

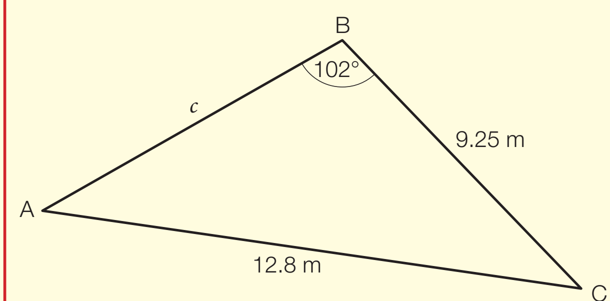
$\frac{8.2}{\sin 64°}=\frac{c}{\sin 48°}$

$c=\frac{8.2\times\sin 48°}{\sin 64°}$

$c=6.78\text{ cm}$

Worked example 15.8

Q. Find the size of angle $B\hat{A}C$.



There is no right angle, so you need to use the sine rule.

Determine which pair of ratios you need.

Cross multiply to solve for sin A.

Use $\sin^{-1}$ key to get the angle in degrees.

A. $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

$\frac{\sin A}{9.25}=\frac{\sin 102°}{12.8}=\frac{\sin C}{c}$

$\frac{\sin A}{9.25}=\frac{\sin 102°}{12.8}$

$\sin A=\frac{\sin 102°\times 9.25}{12.8}=0.70686\ldots$

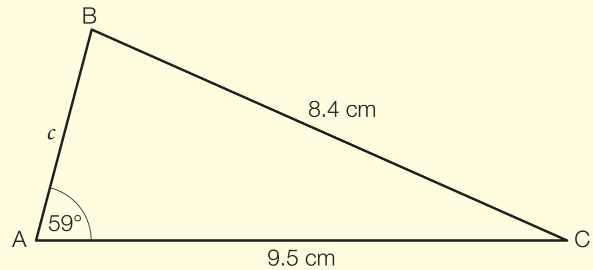
$B\hat{A}C=45.0°$

exam tip

This is called the 'ambiguous case'. It only occurs when you are using the sine rule and have to find an angle larger than the one given. As the second angle can have two different values, the triangle can have two different shapes. The ambiguous case will not be examined.

Worked example 15.9

Q. Find the size of angle $A\hat{B}C$.



A.

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

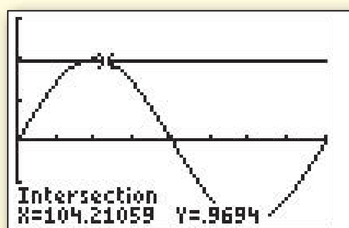
$$\frac{\sin 59°}{8.4}=\frac{\sin B}{9.5}=\frac{\sin C}{c}$$

$$\frac{\sin 59°}{8.4}=\frac{\sin B}{9.5}$$

$$\sin B=\frac{9.5\times\sin 59°}{8.4}=0.9694\ldots$$

$$A\hat{B}C=75.8°$$

If $a > b$ then angle A > angle B. In this example, you are using the sine rule but have been given the size of an angle that is **smaller** than the angle you are trying to find, i.e. $a < b$. This has an interesting result, as you will see at the end.



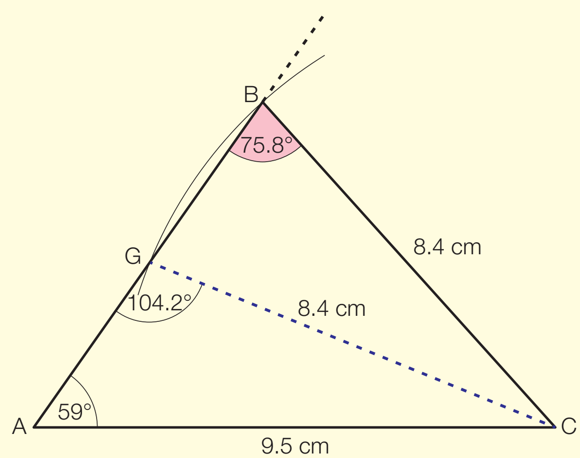
The triangle ABC shows the angle B as an acute angle (and this is what the calculation has produced). But if you draw the sine curve on your GDC, you will see that there are two alternative values of x for which $\sin x = 0.9694$; these are $x = 75.8°$ and $x = 104.2°$.

Triangle ABC could have angles of:

$A=59°, B=75.8°$ and $C=45.2°$

or

$A=59°, B=104.2°$ and $C=16.8°$



If you draw the diagram accurately, with the line AB at an angle of 59° to the line AC at A, then place your compass point at C and set it to 8.4 cm, you will find that it cuts the line AB in two places, hence the two possible solutions.

Exercise 15.4

hint

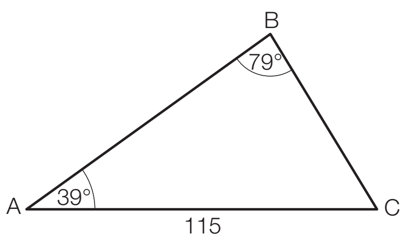
Sketch the diagrams.

1. Some of the dimensions of triangle ABC are given in the table below. The notation corresponds to that used for the sine rule, with the angles labelled as A, B and C. The corresponding opposite sides are labelled as a, b and c. Calculate the value of the missing values, x in each case.

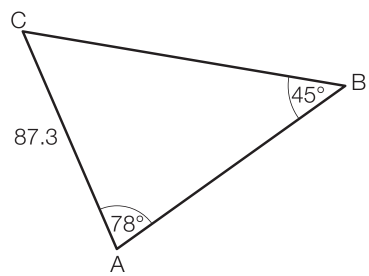
	a	b	c	A	B	C
(a)		137.3	210.3		31.7	x
(b)	x		39.1	44.5		50.7
(c)		x	138.3		10.9	30.6
(d)	140.1	103.1		x	20.5	
(e)	135.1		x	59.9		74.8
(f)	90	71.1		88.6	x	

2. Calculate the length of side BC in each of the following triangles:

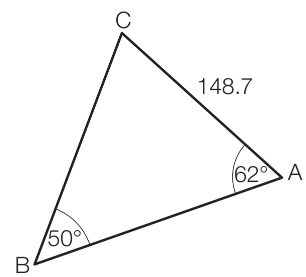
(a)



(b)

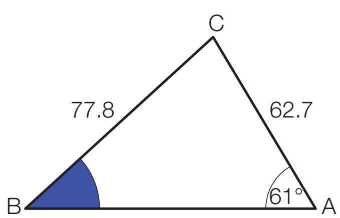


(c)

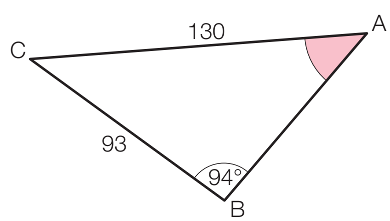


3. Find the size of the shaded angle in each of the following triangles:

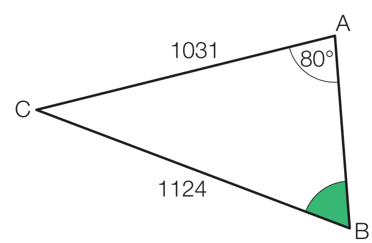
(a)



(b)



(c)



4. In each of the following triangles ABC you are given the lengths of AB and BC. You are also given the size of $B\hat{A}C$. Find the size of $A\hat{C}B$ in each case.

	AB	BC	$B\hat{A}C$
(a)	78.1 cm	140 cm	108°
(b)	108 mm	130 mm	80°
(c)	51 m	65.2 m	62°
(d)	30 km	49.5 km	98°
(e)	42.72 cm	45.28 cm	83.5°

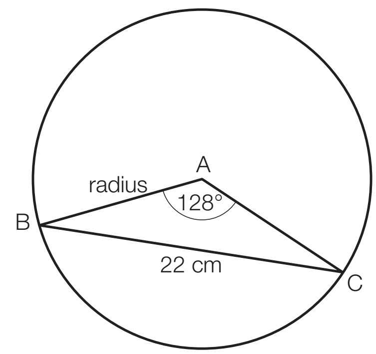
5. In each of the following triangles XYZ, you are given the sizes of $Y\hat{X}Z$ and $X\hat{Y}Z$ and the length of side YZ. Find the length of XZ in each case.

	$Y\hat{X}Z$	$X\hat{Y}Z$	YZ
(a)	85.7°	20.2°	105 cm
(b)	106.5°	35°	120.4 mm
(c)	64.8°	74.1°	50 m
(d)	49.6°	36.7°	30.4 m
(e)	41.8°	39°	42.7 km

hint

'Subtends an angle of 128°' means that the angle between AB and AC or the angle the chord makes at the centre of the circle is 128°.

6. A chord subtends an angle of 128° at the centre of a circle. Given that the length of the chord is 22 cm, find the radius of the circle.

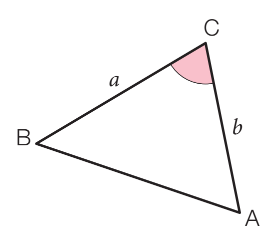


7. From the balcony on the 6th floor of a building, the angle of depression of the top of a tree is 38°. From the balcony on the 12th floor the angle of depression increases to 64°. Given that the distance between the two balconies is 18 m, find the distance between the 12th floor balcony and the top of the tree.

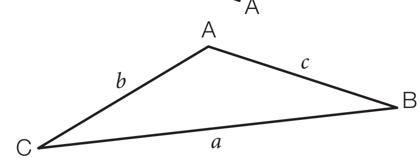
15.5 The cosine rule

If a problem gives you values for:

two sides and the angle between them,



or three sides and no angle,



then you cannot use the sine rule, because you will not be able to fill in a complete ratio with values that you know.

Instead, you can use the cosine rule:

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}\right\}\text{ to find the length of a side,}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{to find the size of an angle.}$$

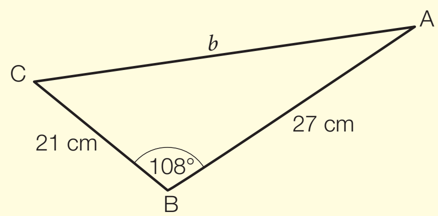
hint

Look for the pattern in all three formulae, and you will find it easy to remember them.

Does this formula look familiar? What is the value of cos 90°?

Worked example 15.10

Q. Find the length AC.



We have a known angle between two known lengths, so use the cosine rule.

A. $b^2 = a^2 + c^2 - 2ac\cos B$

$= 21^2 + 27^2 - 2 \times 21 \times 27\cos 108°$

$= 1520.425\ldots$

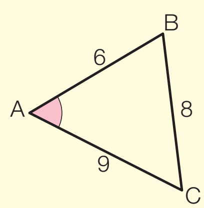
Remember to use the square root key ($\sqrt{\ }$) to find b.

$b = \sqrt{1520.425}$

$b = 39.0$ cm

Worked example 15.11

Q. Find the size of angle A.



You know three lengths and need to find an angle, so use the cosine rule.

A. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{9^2 + 6^2 - 8^2}{2 \times 9 \times 6}$

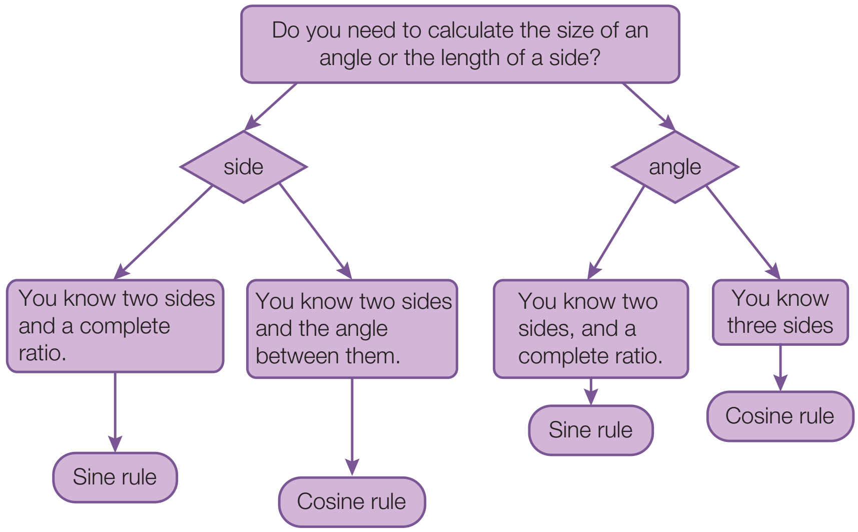
Use the $\cos^{-1}$ key.

$\cos A = 0.4907\ldots$

$A = 60.6°$

Summary

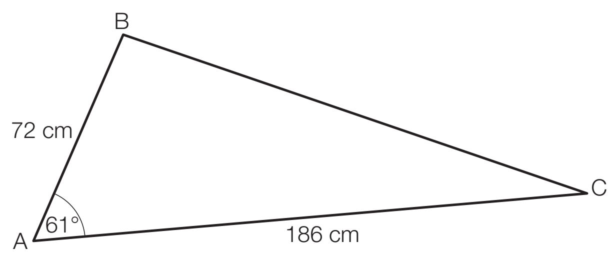
This flow chart can help you to choose the correct trigonometric rule for any given problem.



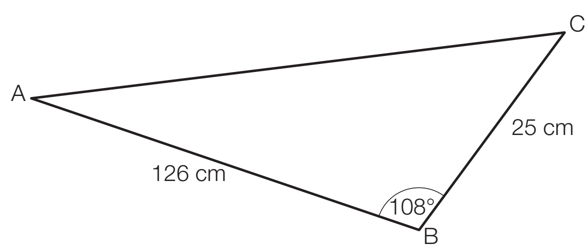
Exercise 15.5

1. Find the length of the side opposite the given angle in each of the following triangles:

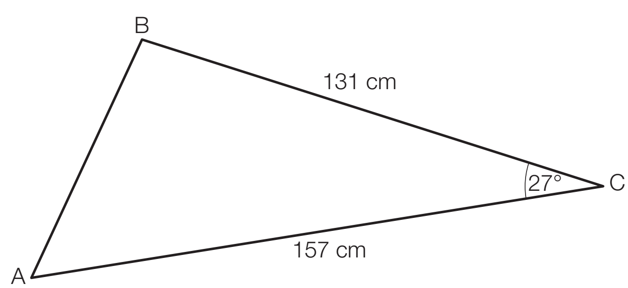
(a)



(b)

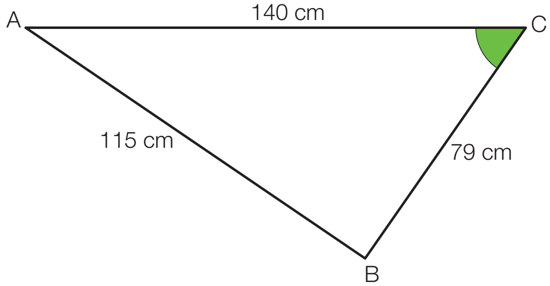


(c)

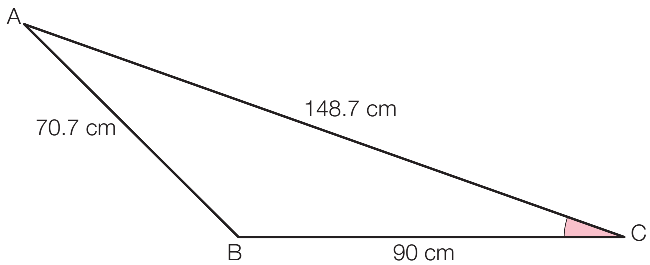


2. Calculate the size of the shaded angle in each of the following triangles:

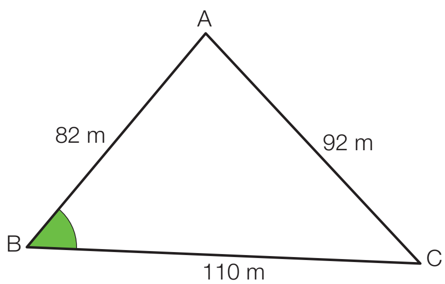
(a)



(b)

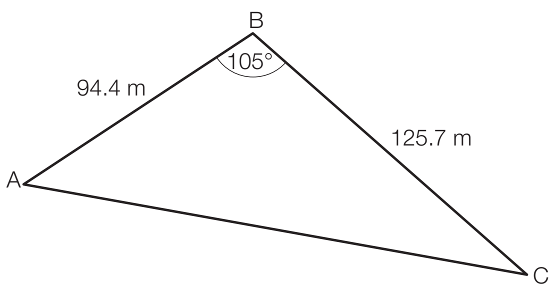


(c)

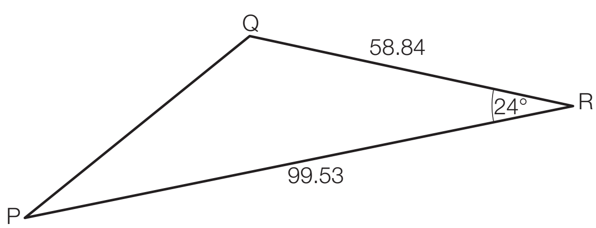


3. Use the cosine rule to find the lengths of all missing sides and the sizes of all missing angles in the following triangles:

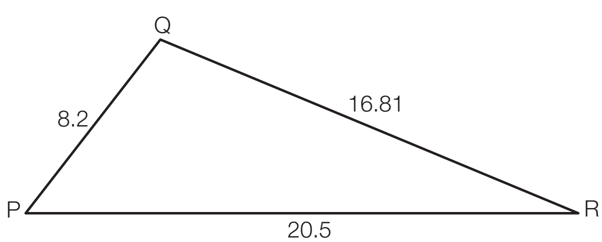
(a)



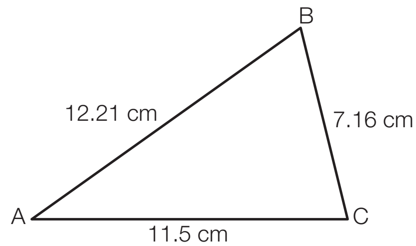
(b)



(c)



(d)

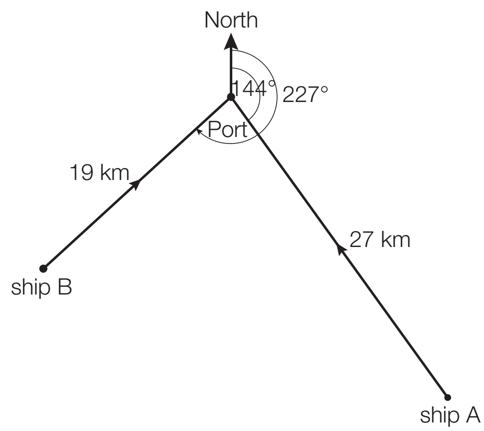


4. Three houses P, Q and R are located in the same neighbourhood. The distance of P from Q is 270 m and the distance of R from Q is 236 m. The angle $P\hat{Q}R$ is 111°. Find the distance between P and R.

hint

For an explanation about angles relative to 'North', see Learning links 15A on page 460.

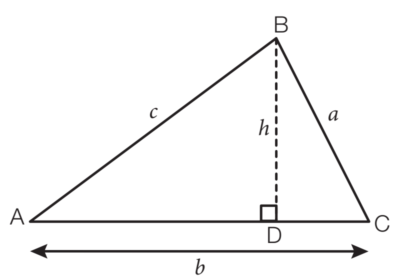
5. Two ships are approaching the same port. Ship A is 27 km away from the port and its path makes an angle of 144° from North. Ship B is 19 km away from the port and its path makes an angle of 227° from North. Find the distance between the two ships.



15.6 Area of a triangle

It is also possible to find the area of a triangle without first having to work out the height of the triangle.

We know that area of a triangle $= \frac{1}{2}$ base $(b) \times$ height (h).



But, using triangle BDC, $\sin C = \frac{h}{a}$ and so $h = a \times \sin C$,

so the formula can be re-written as: area $= \frac{1}{2}\, b \times a \times \sin C$.

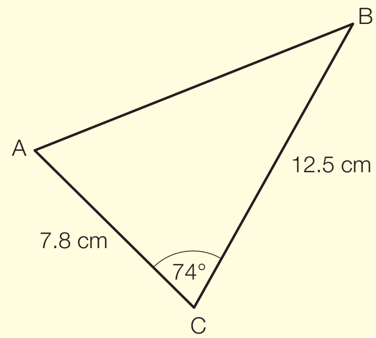
hint

This formula always uses two sides and the angle that is between them. This angle is known as an **included angle**.

$a = \pi r^2$ Area of a triangle $= \frac{1}{2} ab \sin C$

Worked example 15.12

Q. Find the area of triangle ABC.



The perpendicular height of the triangle is not given, but we do have two lengths and an included angle so use the formula $\frac{1}{2}ab\sin c$.

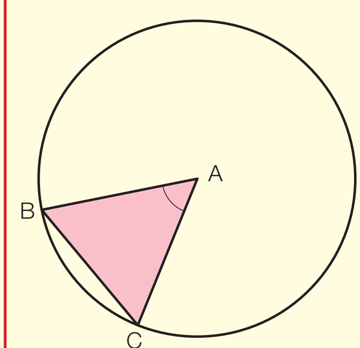
A. $\text{Area} = \frac{1}{2}ab\sin C$

$\text{Area} = \frac{1}{2} \times 7.8 \times 12.5 \times \sin 74°$

$\text{Area} = 46.9\text{ cm}^2$

Worked example 15.13

Q. This circle has a radius of 5 cm and the area of the triangle is 11 cm². Find the size of the acute angle $B\hat{A}C$.



AB and AC are radii of the circle, so we know they are both 5 cm long. $B\hat{A}C$ is the included angle between AB and AC, so we can use the formula Area = $\frac{1}{2}ab\sin c$ to calculate the value of $B\hat{A}C$.

A. $AB = AC = 5\text{ cm}$

$\text{Area} = \frac{1}{2}bc\sin A$

$11 = \frac{1}{2} \times 5 \times 5 \times \sin A$

$22 = 25\sin A$

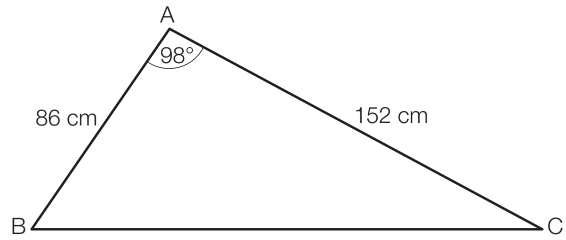
$\sin A = \frac{22}{25} = 0.88$

$B\hat{A}C = 61.6°$

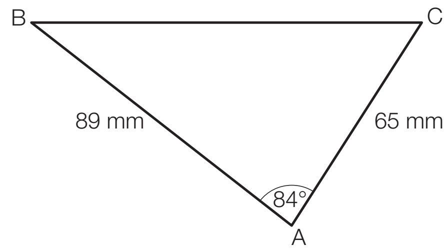
Exercise 15.6

1. Calculate the area of the following triangles:

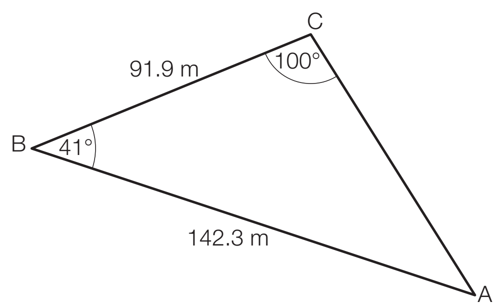
(a)



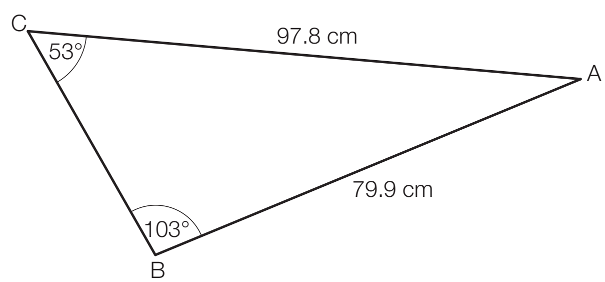
(b)



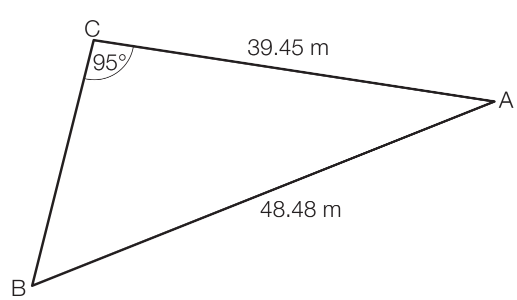
(c)



(d)



(e)



15.7 Constructing labelled diagrams

Practical problems are often expressed in words, which means you should sketch your own diagram and work out which formula you need to use.

When you draw the diagram for a problem, it is important that you:

- read the question more than once, and make sure that you understand it
- draw the diagram large enough; don't try to fit all the information into one small rough sketch
- draw the diagram in pencil; you may need to start again, change it or erase something
- label the diagram clearly and include all the information
- add any extra lines or angles that you are asked to find.

Practical problems set in the examinations are likely to use more than one of the triangle formulae. For every solution it is important to make sure that you have all the information written on your diagram so that you can make the correct decision about which formula to use.

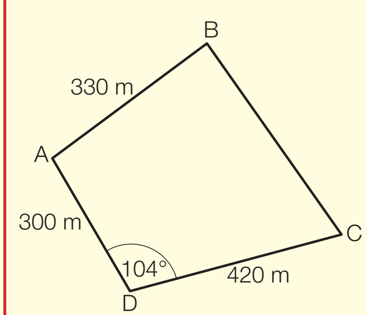
Worked example 15.14

Q. Farmer Giles has a four-sided field, ABCD. He decides to divide the field into two triangles with a fence from A to C.

(a) What is the length of the fence from A to C?

(b) What is the area of field ACD?

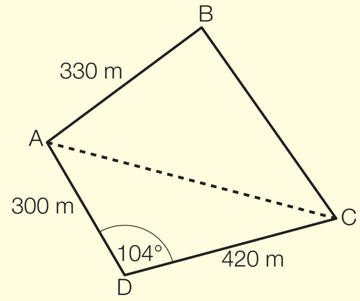
(c) What is the size of angle DCA?



continued . . .

Draw in the line AC to break the shape into two triangles.

A. (a)



There is no right angle, but you are given two sides and the angle between them, so use the cosine rule.

$d^2 = a^2 + c^2 - 2ac\cos D$

$AC^2 = 300^2 + 420^2 - 2 \times 300 \times 420 \times \cos 104°$

$AC^2 = 327364.3\ldots$

$AC = 572\text{ m}$

Use the area formula.

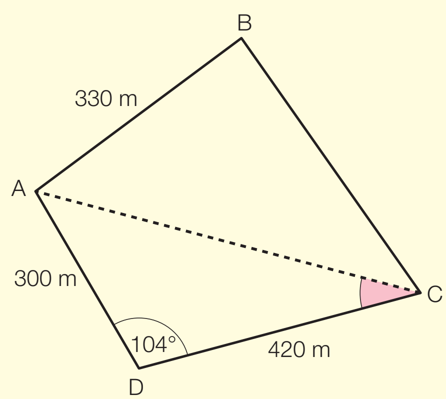
(b) $\text{Area} = \frac{1}{2}ac\sin D$

$\text{Area} = \frac{1}{2} \times 300 \times 420 \times \sin 104°$

$\text{Area} = 61129\text{ m}^2$

$\text{Area} \approx 61100\text{ m}^2$

(c)



You know the size of one angle and the length opposite it, and need to find an unknown angle opposite a known length, so you can use the sine rule to calculate $D\hat{C}A$.

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\frac{\sin C}{c} = \frac{\sin D}{d}$

The value of *d* is from part (a).

$\frac{\sin C}{300} = \frac{\sin 104°}{572}$

$\sin C = \frac{300 \times \sin 104°}{572}$

$\sin C = 0.5088\ldots$

$C = 30.6°$

hint

For an explanation about angles 'from North', see Learning links 15A on the next page.

Worked example 15.15

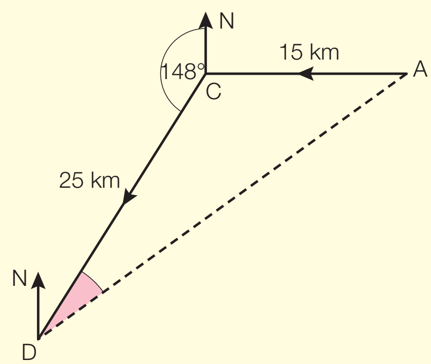
Q. A small plane flies due west for 15 km, then turns through an angle of 148° *anticlockwise* from North and flies another 25 km.

(a) How far is the plane from its starting point?

(b) On the return journey, the plane flies on a straight path, direct to the starting point. What angle does its return flight make with the North line?

A. (a)

Sketch the diagram, making sure that you include all the information given in the question. You need to calculate the length of AD.



Calculate angle $A\hat{C}D$ using the angle relationship 'angles around a point add up to 360°'. The North line is perpendicular to the line AC, so makes an angle of 90°.

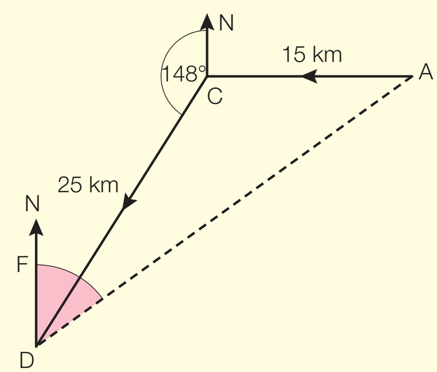
$A\hat{C}D = 360 - 148 - 90 = 122°$

You now know the length of two sides and the size of their included angle, so you can use the cosine rule.

$AD^2 = 15^2 + 25^2 - 2 \times 15 \times 25 \times \cos 122°$

$AD = 35.3\text{ km}$

(b)



You can calculate angle $A\hat{D}C$, using the sine rule.

$$\frac{\sin D}{15} = \frac{\sin 122°}{35.3}$$

$A\hat{D}C = 21.1°$

If we label the North line at D with the letter F, we create the angle $C\hat{D}F$. The line CD is a transversal to two parallel lines (the two North lines) so we use the angle relationship 'co-interior angles add up to 180°' to calculate $B\hat{M}C$.

Angle $= 180° - 148° = 32°$

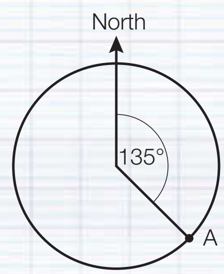
So, angle $F\hat{D}A = 32 + 21.1 = 53.1°$

The return flight makes an angle of 53.1° with the North line.

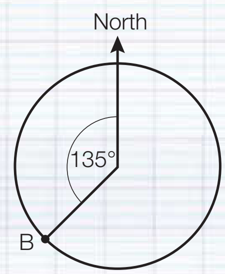
Learning links

15A Angles 'from North'

Directions can be measured by reference to North, South, East or West, or by referring to the North only. When an angle is 'from North', it is measured clockwise from a vertical line denoted by North or N. When this happens, the term 'bearings' is used. Bearings are not included in the syllabus so you will not need to know them for the examinations. You **should** be aware however what is meant by 'clockwise from North' and 'anticlockwise from North'.



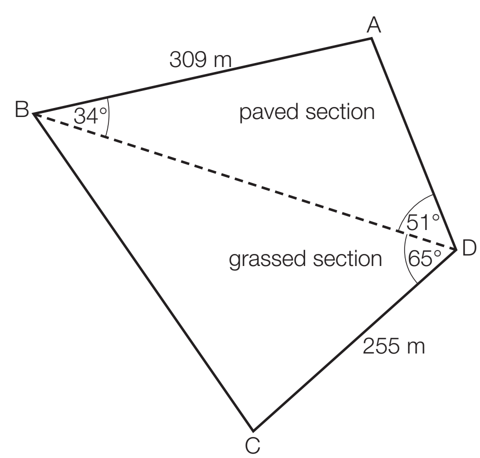
In this example, A is 135° *clockwise* from North.



In this example, B is 135° *anticlockwise* from North.

Exercise 15.7

1. ABCD is the plan of a school playground. It has two sections — a paved area and a grassed section.



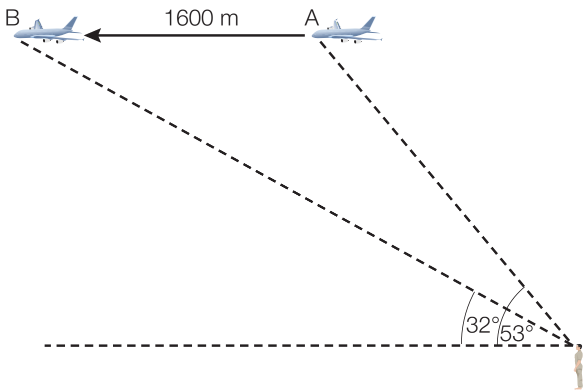
(a) Calculate the length of the boundary line between the two sections of the playground.

(b) Find the area of the paved section.

(c) Calculate the area of the grassed section.

(d) Find the **perimeter** of the playground.

2. When Amanda first sighted a plane at point A the angle of elevation was 53°. After flying a further 1600 m to point B the angle of elevation reduced to 32°. Calculate the distance between Amanda and the plane at point B.

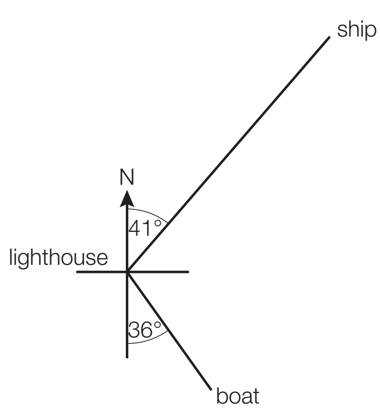


3. A lighthouse stands on a cliff. The angle of elevation of the bottom B of the lighthouse tower from a yacht is 46°. From the same yacht the angle of elevation of the top T of the lighthouse is 58°. Given that the height of the lighthouse is 35 metres, find the height of the cliff above sea level.

4. Three cities A, B and C form a triangle. The distances between the cities, AB, BC and AC, are 186.1 km, 208.6 km and 314 km respectively. Sketch a diagram to represent the above information and hence find the sizes of the three angles of the triangle.

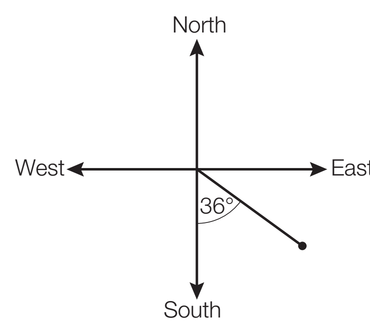
5. Two towns, Silvagrad and Tinburgh, are 208 km apart. Tinburgh is located due east of Silvagrad. A third town, Westown, is located north-east of Tinburgh and 150 km away from Tinburgh. Find the distance of Westown from Silvagrad.

6. John is at the lighthouse. He sights a ship 15 km away at an angle of 41° east from North. At the same time he sights a boat 6 km away, at an angle of 36° east from South. How far away are the boat and the ship from each other?



hint

'36° east from South' would be:



In other words, it is 36° to the right (east) of the 'South' direction.

Summary

You should know:

- what the three different trigonometric ratios in right-angled triangles are:
  - $\sin(\text{e})\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
  - $\cos(\text{ine})\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
  - $\tan(\text{gent})\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
- that the trigonometric ratios can be used to find the length of unknown sides and the size of unknown angles in right-angled triangles
- the definition of angles of elevation and depression
- that when a triangle is not right-angled, different trigonometric ratios are needed:
  - sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ to find the length of a side or

    $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ to find the size of an angle
  - the cosine rule,

    $$\left.\begin{aligned} &a^2 + b^2 + c^2 - 2bc\cos \text{A} \\ &b^2 = a^2 + c^2 - 2ac\cos \text{B} \\ &c^2 = a^2 + b^2 - 2ab\cos \text{C} \end{aligned}\right\} \quad \text{to find the length of a side, or}$$

    $\cos \text{A} = \dfrac{b^2 + c^2 - a^2}{2bc}$ to find the size of an angle — this is useful when you have an included angle.
- the formula for the area of a triangle, when you do not know the perpendicular height, $A = \frac{1}{2}ab\sin \text{C}$
- how to construct labelled diagrams from verbal statements.

Mixed examination practice

Exam-style questions

1. Kitty and Betty are standing on the same side of a radio mast. The angle of elevation of the top of the mast from Kitty is 32°. From Betty's position the angle of elevation of the top of the mast is 43°. Given that the height of the mast is 198 m, find the distance between Kitty and Betty.

2. The horizontal distance between two buildings is 200 m. From the top of the taller building the angle of depression of the top of the shorter building is 28°. From the same point on the taller building the angle of depression of the bottom of the shorter building is 42°. Calculate the heights of both buildings.

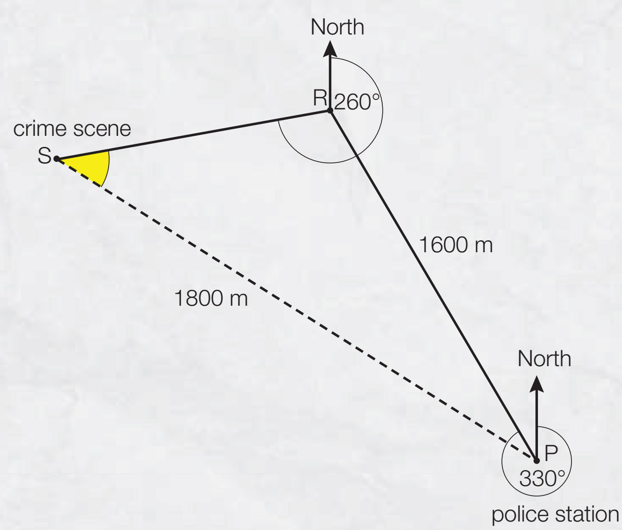
3. Yuri and Yuko are trekking in the forest. They both set off due east. After walking a distance of 2000 m Yuri changes direction and heads due south for another 700 m and stops. Yuko continues to walk another 1000 m due east before she changes direction. She then walks due north for another 1600 m and stops.

   (a) Calculate the distance between the two girls.

   (b) Through what angle must Yuri turn in order to walk directly towards Yuko?

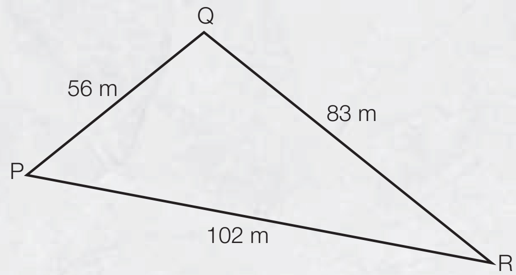
4. A motorcyclist is riding on a straight road towards a tower. From point A on the road, the angle of elevation of the top of the tower is 25°. After he has travelled a further 80 m towards the tower, the angle of elevation of the top of the tower increases to 40°. Find the distance between point A and the top of the tower.

5. A police helicopter departs from the police station, flying on a path 330° clockwise from North. After flying for 1600 m, it receives a message to divert to a crime scene. It then changes direction to fly to the crime scene and its new path is 260° clockwise from North. The crime scene is 1800 m in a diagonal straight line from the police station. The helicopter's route is illustrated below.



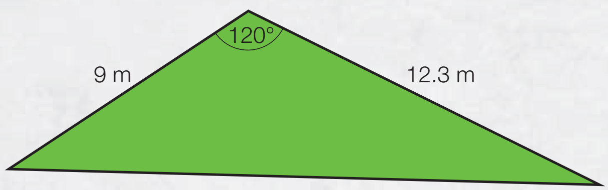
   Calculate the size of angle $R\hat{S}P$.

6. Three friends, Pat, Queenie and Rahina, are picking wildflowers in a field. Pat is 56 m away from Queenie. Queenie (Q) is 83 m from Rahina(R) and Rahina is 102 m away from Pat (P).

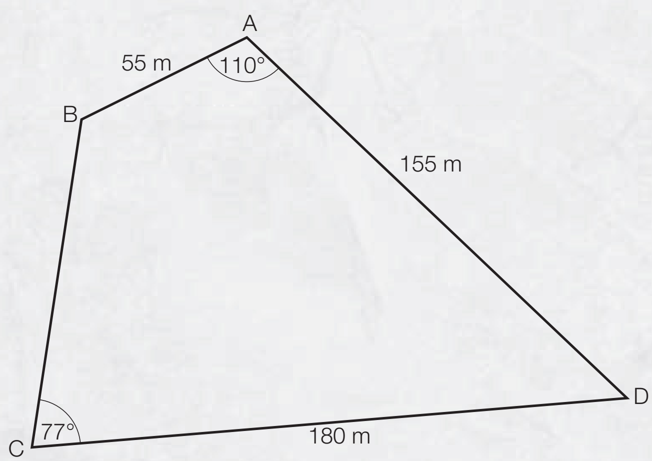


Given that the angle between PQ and North at P is 51°, calculate the value of the reflex angle between RQ and the North direction at R.

7. Jo has been offered a triangular shaped allotment. Two sides measure 9 m and 12.3 m. If the size of the included angle is 120°, calculate the area of the allotment.



8. Mr Bonsu has a piece of farmland which he wants to sell. The land is in the shape of a quadrilateral ABCD. Three sides of the land, AB, AD and CD, have lengths 55 m, 155 m and 180 m respectively. Mr Bonsu wants to estimate the perimeter and the area of the farmland.



(a) Calculate the length of BD.

(b) Work out the magnitude of angle $C\hat{B}D$.

(c) Calculate the length of side BC and hence the perimeter of the farmland.

(d) Work out the area of the farmland.

9. A and B are two points on a level ground. They are located on opposite sides of a tall tree. From point A, the angle of elevation of the top of the tree is 20°. From point B, the angle of elevation of the top of the tree is 30°. The distance between the two points is 70 metres.

Calculate the height of the tree.

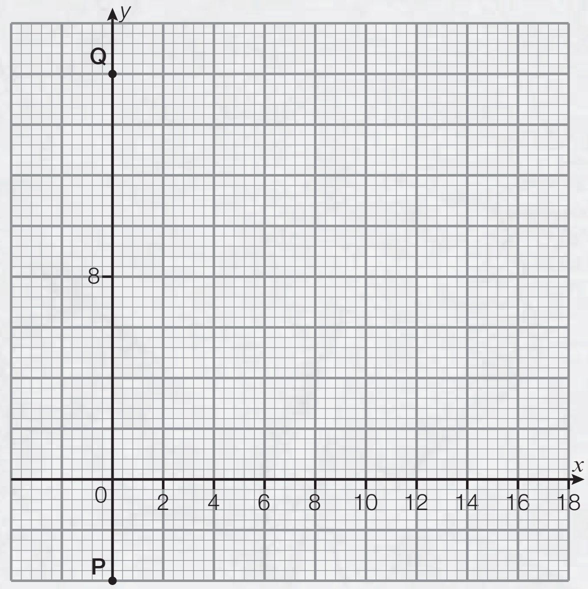
10. Three friends Darya, Maiya and Zeena are standing at three different positions on level ground. Maiya is standing 50 metres due east of Darya. Zeena is standing 30 metres due north of Darya. Maiya and Zeena are standing at two different positions along a straight river bank.

Darya decides to walk towards the river bank, taking the shortest route.

How far will Darya have to walk to get to the river?

Past paper questions

1. Points P(0,–4) and Q(0, 16) are shown on the diagram.



(a) Plot the point R(11, 16).

(b) Calculate angle QPR.

M is a point on the line PR. M is 9 units from P.

(c) Calculate the area of triangle PQM. *[Total 6 marks]*

[May 2006, Paper 1, TZ0, Question 6] (© *IB Organization 2006*)

2. Triangle ABC is drawn such that angle ABC is 90°, angle ACB is 60° and AB is 7.3 cm.

(a) (i) Sketch a diagram to illustrate this information. Label the points A, B, C.
Show the angles 90°, 60° and the length 7.3 cm on your diagram.

(ii) Find the length of BC. *[3 marks]*

Point D is on the straight line AC extended and is such that angle CDB is 20°.

(b) (i) Show the point D and the angle 20° on your diagram.

(ii) Find the size of angle CBD. *[3 marks]*

[Total 6 marks]

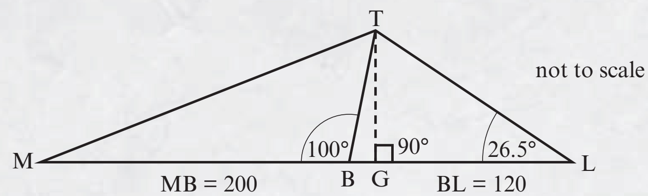
[May 2008, Paper 1, TZ2, Question 3] (© *IB Organization 2008*)

3. An old tower (BT) leans at 10° away from the vertical (represented by line TG).

The base of the tower is at B so that $M\hat{B}T = 100°$.

Leonardo stands at L on flat ground 120 m away from B in the direction of the lean.

He measures the angle between the ground and the top of the tower T to be $B\hat{L}T = 26.5°$.



(a) (i) Find the value of angle $B\hat{T}L$.

(ii) Use triangle BTL to calculate the sloping distance BT from the base, B to the top, T of the tower. *[5 marks]*

(b) Calculate the vertical height TG of the top of the tower. *[2 marks]*

(c) Leonardo now walks to point M, a distance 200 m from B on the opposite side of the tower. Calculate the distance from M to the top of the tower at T. *[3 marks]*

[Total 10 marks]

[**May 2007, Paper 2, TZ0, Question 2(ii)**] (© *IB Organization 2007*)

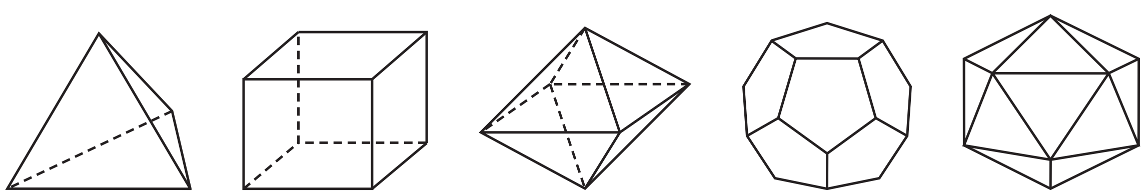
Chapter 16 Geometry of three-dimensional solids

In this chapter you will learn:

- about the geometry of the following three-dimensional solids: cuboid, right prism, right pyramid, cylinder, sphere, hemisphere, cone and combinations of these solids
- how to calculate the distance between two points within a solid, such as the distance between two vertices, between a vertex and a midpoint, or between different midpoints
- how to calculate the size of an angle between two lines or between a line and a plane
- how to calculate the volume and surface areas of the three-dimensional solids defined above.

Plato (429–347 BCE) is generally considered to have been a philosopher, but he also gave his name to the group of regular **solids** called the Platonic solids. These are three-dimensional shapes whose flat faces are regular polygons. In ancient Greece, science, mathematics and philosophy were not separate disciplines; it was considered the mark of an educated person that they were interested in all branches of knowledge.

The five Platonic solids are:

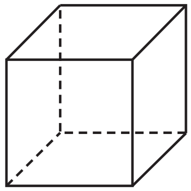


tetrahedron | **cube** | octahedron | dodecahedron | icosahedron

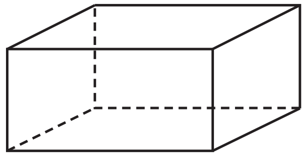
In Mathematics, the word 'solid' is used to describe any three-dimensional shape. A mathematical solid can even be hollow!

There are other solids that are not Platonic. These solids have faces of different shapes within the same solid. For example, a square-based pyramid has a square face as the base and the other four faces are triangles.

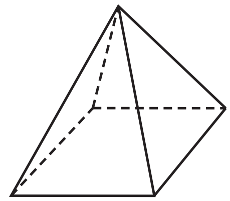
Some solids have edges that are straight lines:



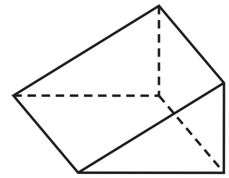
cube



cuboid

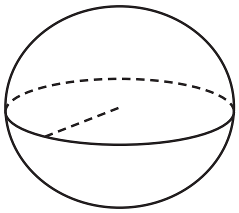


pyramid

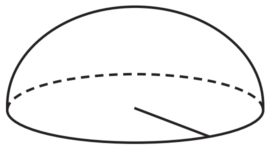


triangular **prism**

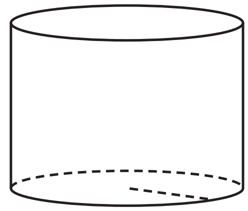
Some solids have curved edges, or a mix of curved and straight edges:



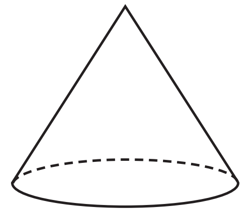
sphere



hemisphere

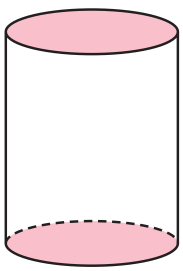


cylinder

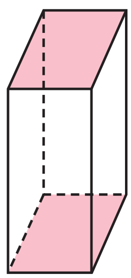


cone

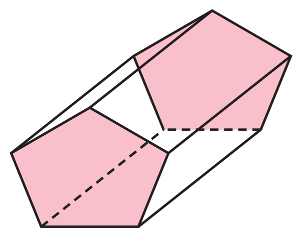
Some solids are called prisms. A prism is a solid object that has two congruent ends and the cross-section is the same shape and area along its whole length.



cylinder

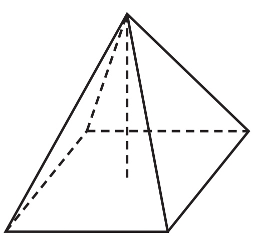


cuboid

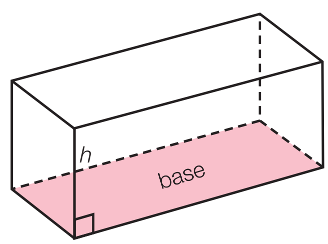


prism with a polygon as its base

The word 'right' is used to describe some solids in which the 'height' makes a right angle with the base.



A **right pyramid** has its apex directly above the centre of its base.

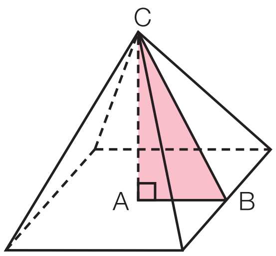


A **right prism** has its base directly below its top surface, and the height makes a right angle with the base.

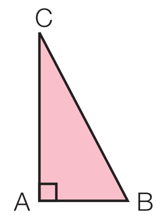
16.1 Finding the length of a line within a three-dimensional solid

Trigonometry and Pythagoras' theorem can be used to solve problems that require you to find the lengths of lines within a solid. These lines may be inside or outside the shape. It can be helpful to think of the solid as being hollow, or to make a model so that you can move it around and understand it better.

In order to do the calculations involved in this sort of problem, you will need to find the triangles within the shape. This is easier to do if you draw these 'calculation' triangles outside the main diagram.



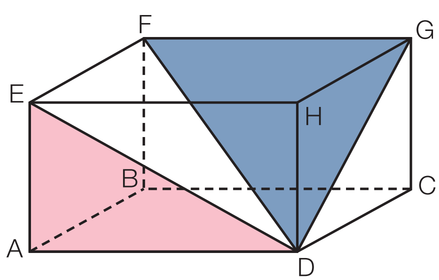
pyramid



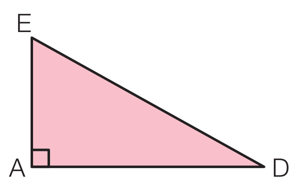
'calculation' triangle

These familiar shapes all contain unexpected right-angled triangles.

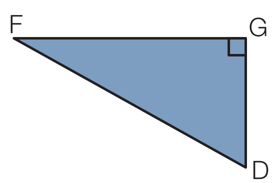
Cuboid



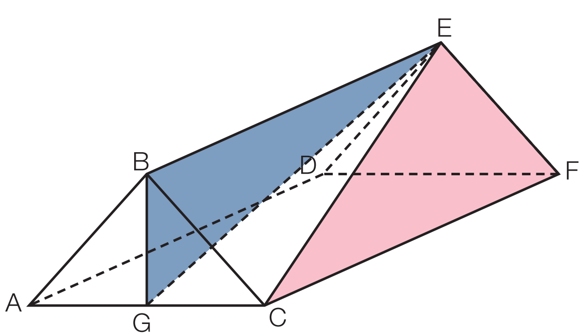
Triangle EAD is right-angled.



So is triangle FGD.

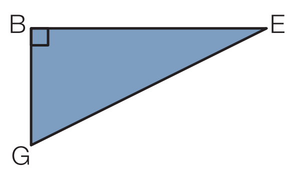


Triangular prism

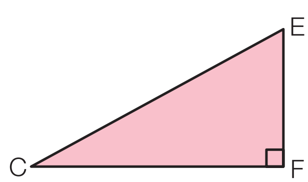


G is the midpoint of AC.

Triangle GBE is right-angled.



So is triangle CFE.

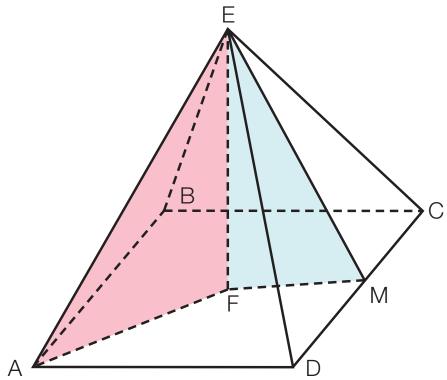


RR ***You learned how to use trigonometry to solve problems in Chapter 15.***

exam tip

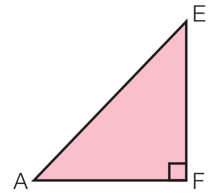
In examinations all questions on solids will be set so that you can answer them using right-angled triangles.

Pyramid

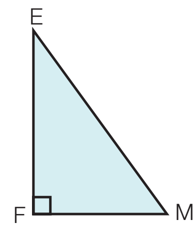


M is the midpoint of CD.

Triangle AFE is right-angled.

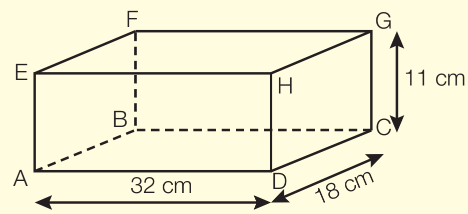


So is triangle EFM.



Worked example 16.1

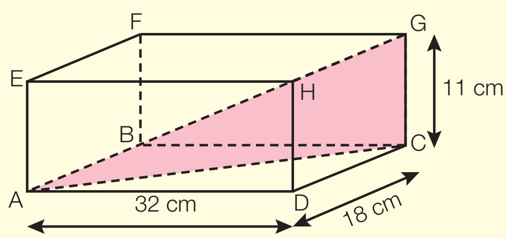
Q. A shoe box is 32 cm long, 18 cm wide and 11 cm high. Calculate the length of the longest rod that will fit inside the box.



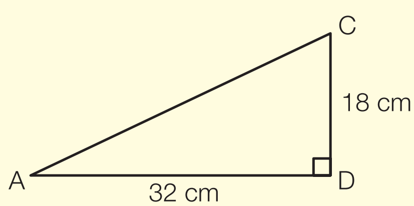
The longest length inside the box is the diagonal from one of the top vertices to the opposite bottom vertex. This is the triangle that you need to work with.

You need the triangle ACD to calculate the length of AC, and then the triangle AGC to calculate the length of AG.

A.



Draw triangle ACD separately first and use Pythagoras' theorem to calculate the length of AC.



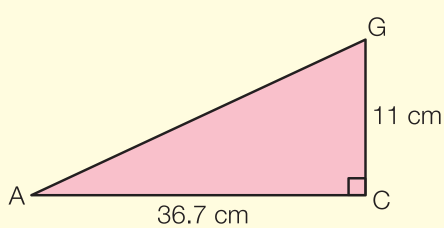
$AC^2 = CD^2 + AD^2$

$AC^2 = 18^2 + 32^2$

$AC = \sqrt{1348}$

$AC = 36.7\text{ cm}$

Then draw triangle ACG and use Pythagoras' theorem to calculate the length of AG.



$AG^2 = AC^2 + CG^2$

$AG^2 = 36.7^2 + 11^2$

$AG = \sqrt{1467.89}$

$AG = 38.3\text{ cm}$

Worked example 16.2

Q. EABCD is a square-based right pyramid, with sides of 12 cm. The height of the pyramid is 9.5 cm.

(a) Calculate the length of EC.

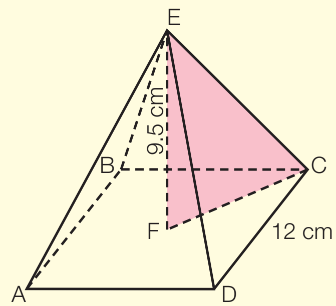
A point, M, is marked half way along one side of the base.

(b) Calculate the length of EM.

(c) Why is the length of EC not the same as EM?

A. (a)

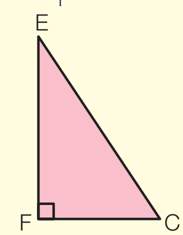
Draw the solid and add in a helpful triangle.



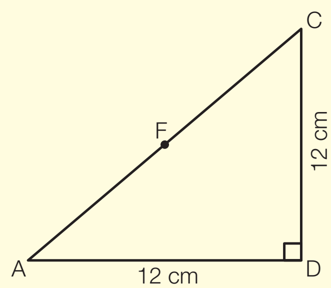
$CD = AD = AB = BC = 12$ cm

$EF = 9.5$ cm.

F is the midpoint of AC.



To find the length FC, you first need to use Pythagoras' theorem to find AC in triangle ACD.



$AC^2 = AD^2 + DC^2$

$AC^2 = 12^2 + 12^2$

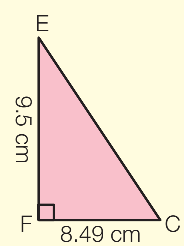
$AC^2 = 144 + 144 = 288$

$AC = \sqrt{288} = 16.97\ldots$

$FC = \frac{1}{2} \times AC$

$FC = 8.49$ cm

To find the length EC, use the length FC and Pythagoras' theorem again.



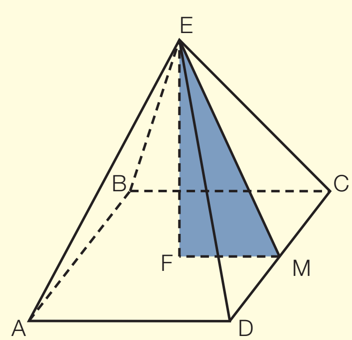
$EC^2 = EF^2 + FC^2$

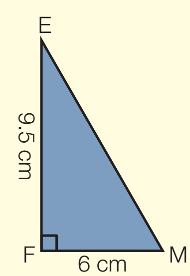
$EC^2 = 9.5^2 + 8.49^2 = 162.25$

$EC = 12.7$ cm

(b)

Add M to the diagram and draw another helpful triangle.





You can chose any edge to mark M on; as all edges are the same length, EM is the same for all four edges. We have chosen M to be the **midpoint** of CD.

continued . . .

$FM = \frac{1}{2}$ of $AD = \frac{1}{2} \times 12$

$FM = 6\text{ cm}$

$EM^2 = EF^2 + FM^2$

$EM^2 = 9.5^2 + 6^2 = 126.25$

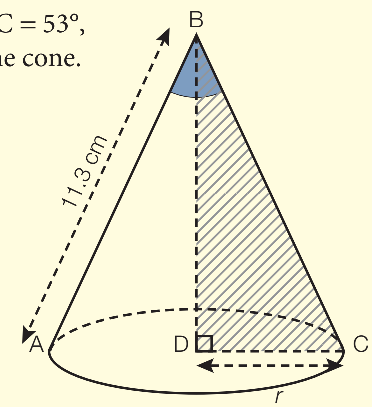
$EM = \sqrt{126.25} = 11.2\text{ cm}$

(c) $EM \neq EC$

The length EC is one of the slanted edges of the pyramid. The length EM is the vertical height of the triangle EDC.

Worked example 16.3

Q. If $AB = 11.3$ cm and $A\hat{B}C = 53°$, calculate the **radius** of the cone.

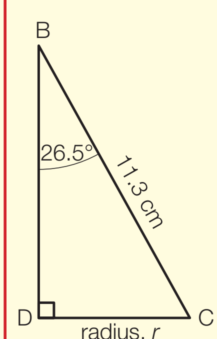


hint

Remember that the radius of a circle is the distance from the centre of the circle to the circumference. It is also half of the diameter.

Sketch the right-angled triangle BDC. Use trigonometry to find the length *r*. You have the opposite and the hypotenuse, so use sin.

A.



$A\hat{B}C = 53°$

$D\hat{B}C = 53° \div 2 = 26.5°$

$\sin 26.5° = \frac{r}{11.3}$

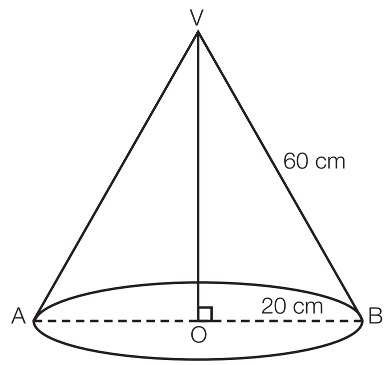
$r = 11.3 \times \sin 26.5°$

$r = 5.04\text{ cm}$

Exercise 16.1

1. VAOB is a right circular cone with radius 20 cm and **slant height** 60 cm.

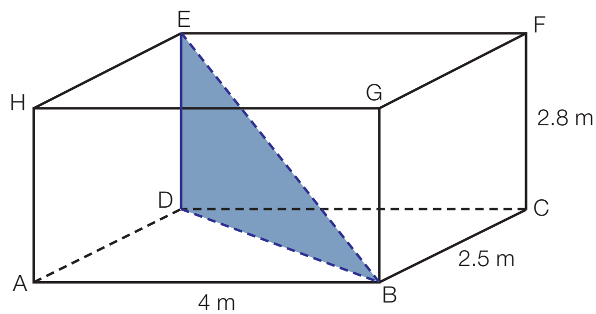
   Calculate the vertical height, VO.



2. ABCDEFGH is a cuboid with dimensions 4 m by 2.5 m by 2.8 m.

   (a) Calculate the length of BD.

   (b) Find the length of the diagonal BE.



3. What is the longest rod that will fit inside a box 20 cm long, 12 cm wide and 16 cm high?

hint

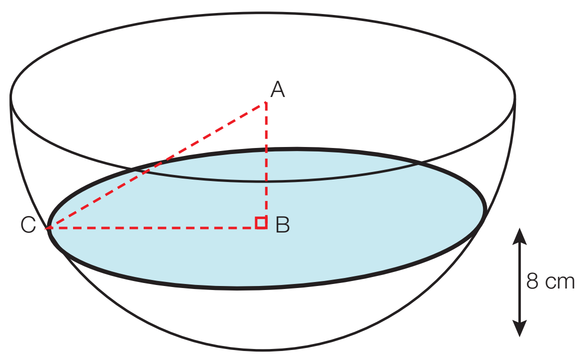
A hemisphere is exactly half a sphere.

4. The diagram below shows a hemispherical bowl of radius 15 cm. Some liquid is poured into the bowl to a height of 8 cm with a circular surface of radius BC.

   (a) State the length of AC.

   (b) Write down the length AB.

   (c) Hence find the length of BC.

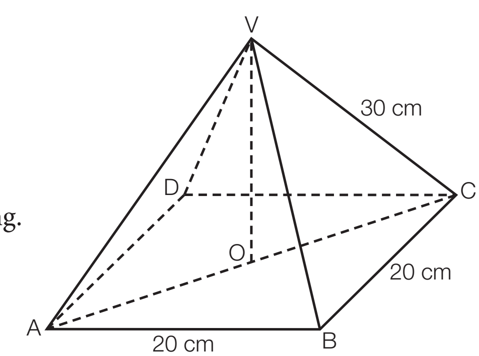


5. VABCD is a right pyramid on a square base. VO is the vertical height of the pyramid.

   The length of each side of the square base is 20 cm, and the slant height (VC) is 30 cm long.

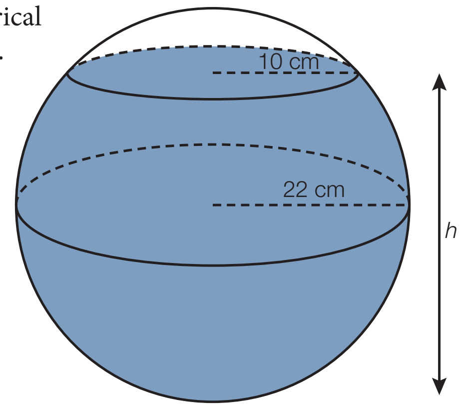
   (a) Calculate the length of AC.

   (b) Calculate the vertical height of the pyramid, VO.



6. Liquid is poured into a spherical container to a height of h cm. The radius of the sphere is 22 cm. The top of the liquid is a circle of radius 10 cm.

Work out the height of the liquid, h.



hint

Remember that a cross-section is a vertical slice through the length of a solid; and an isosceles triangle has two sides of equal length and two angles of equal size. The angle sum of all triangles is 180°.

7. The diagram shows farmer Neil's garden shed. The cross-section of the shed is a pentagon. BCD and IHG are **isosceles triangles**. AB = ED = 2 m. AE = BD = 3 m. EF = DG = 5 m. The vertical height of both C and H above the ground is 4.5 m.

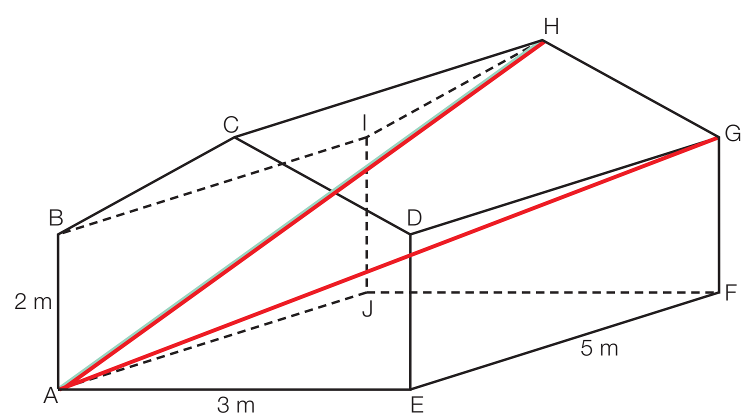
(a) Calculate the length of AF.

K is the midpoint of JF.

(b) Calculate the length of AK.

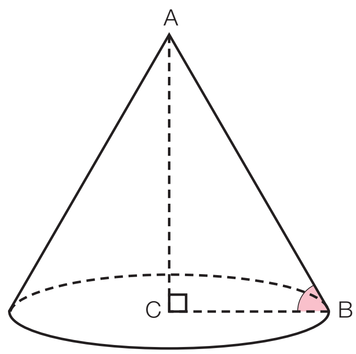
Farmer Neil fits two straight wooden poles, AH and AG, into his shed.

(c) Which of the two poles is longer and by how much?

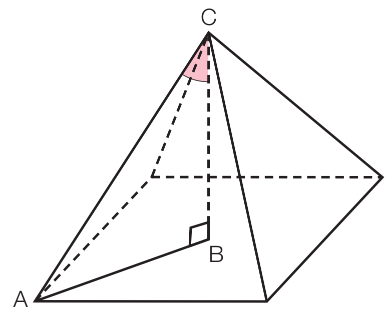


16.2 Finding the size of an angle in a three-dimensional solid

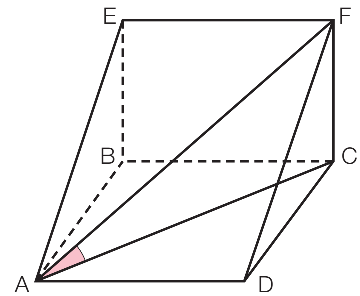
Finding angles in three dimensions uses similar techniques to finding lengths. You will need to identify a right-angled triangle within the solid and mark in the angle that you have been asked to find. Again, it will be easier to do the calculations if you sketch extra diagrams.



Angle $\hat{ABC}$ is the angle between the base of the cone and its slant edge, AB.

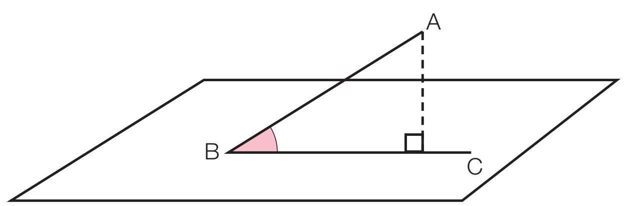


Angle $\hat{ACB}$ is the angle between the height of the pyramid and the edge CA.



Angle $\hat{FAC}$ is the angle between the line AF and the base of the prism, ABCD.

The angle between a line and a plane (a flat surface) is shown in this diagram. It is the angle between the line AB and the line BC that lies on the plane. The perpendicular line has been drawn to create a right-angled triangle.



Worked example 16.4

In every part of the question, the right-angled triangle has been identified and then sketched as a separate diagram. Make sure that you have copied all the letters from the three-dimensional diagram onto your new diagram correctly. Once you have done this, complete your calculations using Pythagoras' theorem or trigonometry.

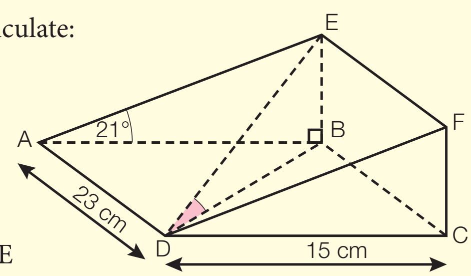
Q. For this triangular prism, calculate:

(a) the height EB

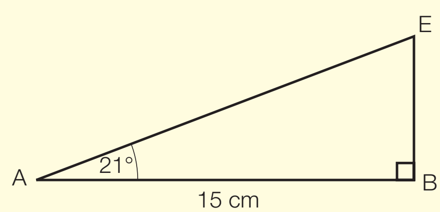
(b) the length of the line DB

(c) the length of the line DE

(d) the angle that the line DE makes with the base of the prism, ABCD.



A. (a)



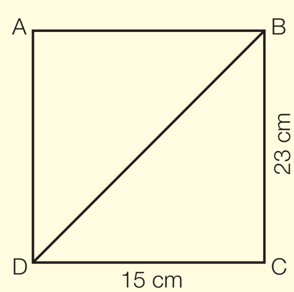
$\tan 21° = \frac{EB}{15}$

$EB = 15 \times \tan 21°$

$EB = 5.76$ cm

(b)

This is the base of the solid.



$DB^2 = 15^2 + 23^2$

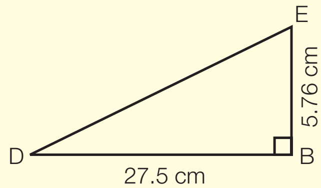
$= 754$

$DB = \sqrt{754}$

$DB = 27.5$ cm

continued . . .

(c)

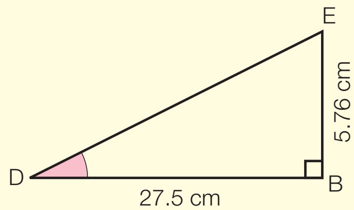


$DE^2 = 5.76^2 + 27.5^2$

$= \sqrt{789.42}$

$DE = 28.1$ cm

(d)



$\tan E\hat{D}B = \frac{5.76}{27.5}$

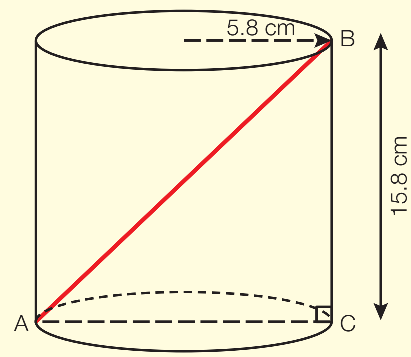
$E\hat{D}B = 11.8°$

Worked example 16.5

Q. A cylindrical drum contains a stick that fits the diagonal exactly. If the radius of the drum is 5.8 cm and the height is 15.8 cm, calculate:

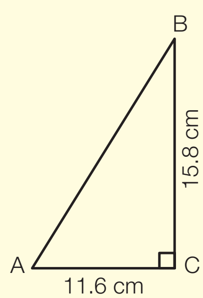
(a) the length of the stick

(b) the angle that the stick makes with the side of the drum.



A. (a)

The length of the stick is given by the line AB.



If the radius of the cylinder is 5.8 cm, the diameter of the cylinder is 11.6 cm.

$AB^2 = 15.8^2 + 11.6^2$

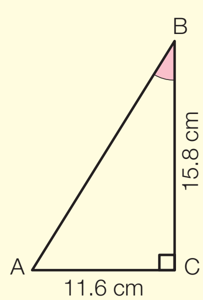
$= \sqrt{384.2}$

$AB = 19.6$ cm

(b)

You now know all three sides of the triangle ABC, so you can use any trigonometric ratio to work out the angle $A\hat{B}C$.

It is more accurate to use the original measurements though.

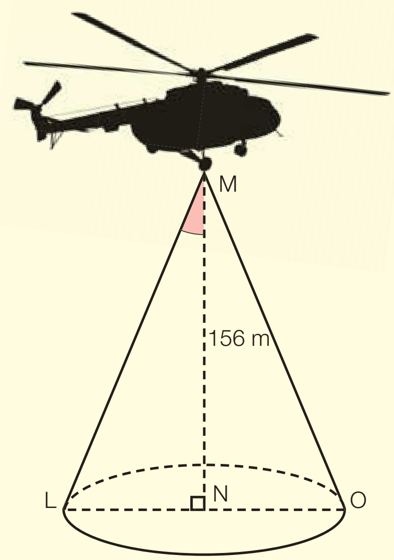


$\tan A\hat{B}C = \frac{11.6}{15.8}$

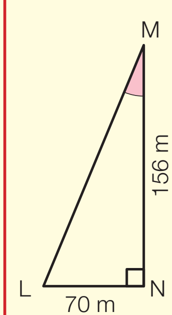
$A\hat{B}C = 36.3°$

Worked example 16.6

Q. A rescue helicopter is hovering 156 m above the sea. It is dark and the helicopter is shining a light down onto the surface of the sea. If the diameter of the cone formed by the beam of the light is 140 m, what is the angle at the apex of the cone?



A.



Use the right-angled triangle LMN. You need to find the angle $L\hat{M}N$ so use tan because you know the length of the opposite and the adjacent.

The diameter of the cone is 140 m.

The radius of the cone $= 140 \div 2 = 70$ m.

$$\tan L\hat{M}N = \frac{70}{156}$$

$L\hat{M}N$ is only half the angle at the apex of the cone, so multiply your answer by 2. Remember not to round your answer too soon.

$L\hat{M}N = 24.1666°$

So the angle $L\hat{M}O = 2 \times 24.1666°$.

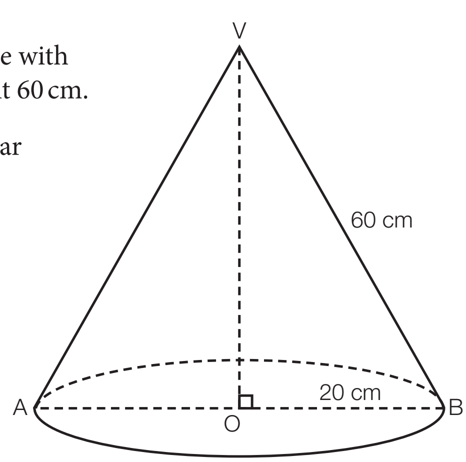
$L\hat{M}O = 48.3°$

Exercise 16.2

1. VAOB is a right circular cone with radius 20 cm and slant height 60 cm.

   (a) What is the perpendicular height of the cone?

   (b) Calculate angle $A\hat{V}B$.

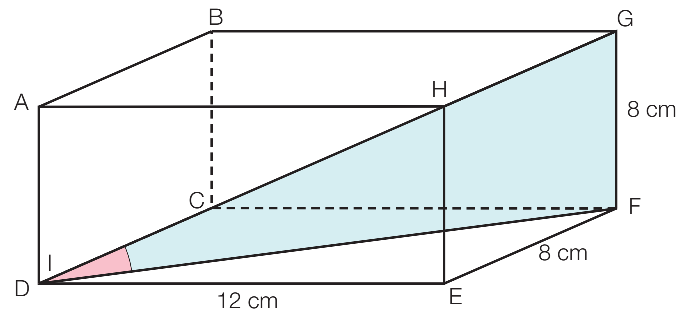


2. ABCDEFGH is a cuboid 12 cm long, 8 cm wide and 8 cm high.

(a) Calculate the length of DF.

(b) Find the length of the diagonal DG.

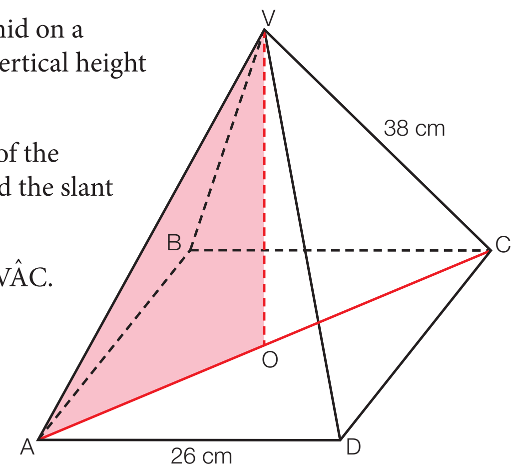
(c) Calculate angle $G\hat{D}F$.



3. VABCD is a right pyramid on a square base. VO is the vertical height of the pyramid.

The length of each side of the square base is 26 cm, and the slant height (VC) is 38 cm.

Find the value of angle $V\hat{A}C$.



4. ABCDEF is a triangular prism.

AB = ED = FC = 80 cm.

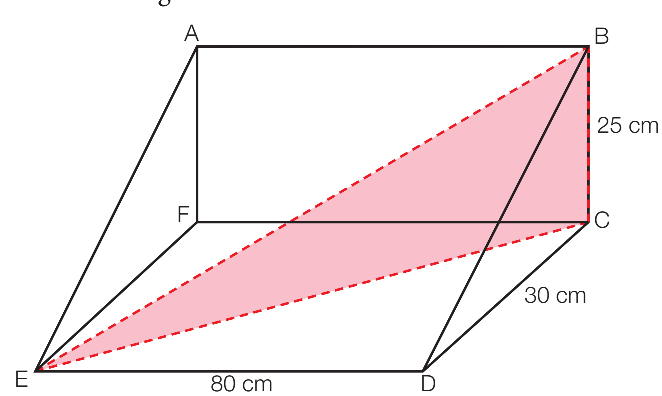
BC = AF = 25 cm and DC = EF = 30 cm.

Calculate:

(a) the length of EC

(b) the length of EB

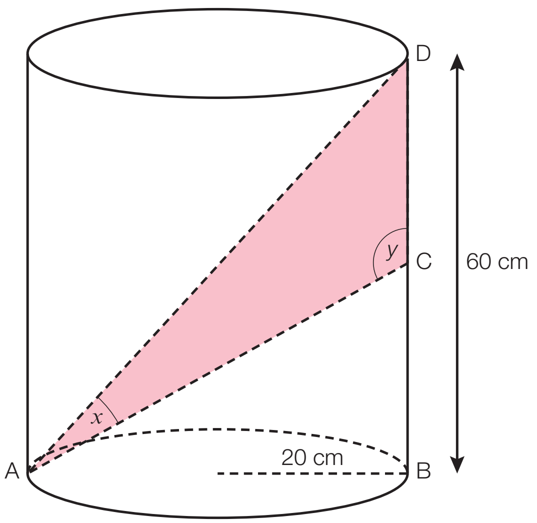
(c) the size of angle $B\hat{E}C$.



5. The diagram below shows a cylinder with radius 20 cm.

The height of the cylinder BD is 60 cm. C is the midpoint of BD.

Calculate the values of angles *x* and *y*.



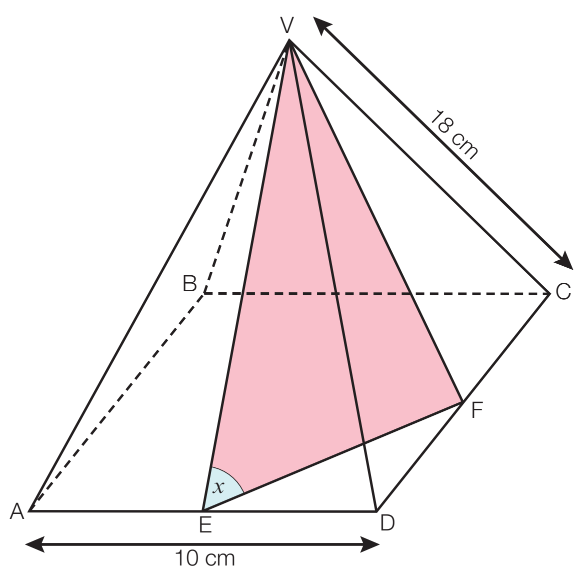
6. VABCD is a right pyramid. The base of the pyramid is a square of side 10 cm. The slant height of the pyramid is 18 cm. E and F are the midpoints of AD and DC respectively.

(a) Calculate the length of:

(i) VE

(ii) EF.

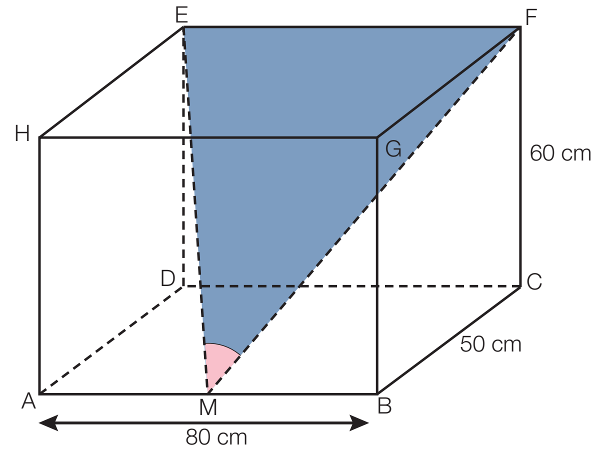
(b) Work out the size of angle *x*.



7. ABCDEFGH is a cuboid of length 80 cm, width 50 cm and height 60 cm. M is the midpoint of AB.

   (a) Calculate the lengths of MF and ME.

   (b) Work out the value of angle $E\hat{M}F$.



8. ABCDEF is a triangular prism. M is the midpoint of AC.

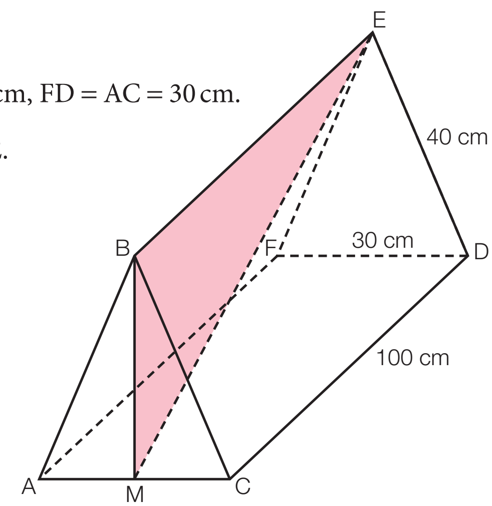
   CD = AF = BE = 100 cm,

   ED = EF = AB = CB = 40 cm, FD = AC = 30 cm.

   (a) Find the length of ME.

   (b) Calculate angle $M\hat{E}B$.

   (c) Calculate angle $M\hat{D}B$.

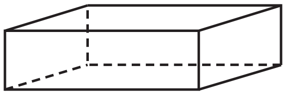
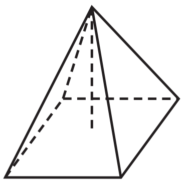
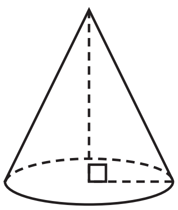
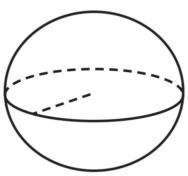
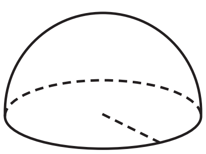
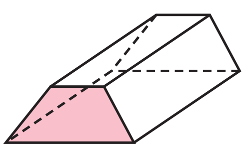
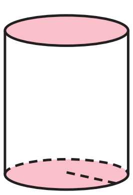


16.3 Calculating volumes and surface areas of three-dimensional solids

On the next page is a table listing some solids and the formula for calculating the volume and surface area. Not all of the formulae are in the Formula booklet, so make sure you know which ones you need to learn.

The solid shapes in the table on the next page are important in architecture, in design, and in many other aspects of everyday life.

The geometry of the surface of a sphere is not the same as that of a flat plane. Are you sure that the angles of a triangle add up to 180° on every surface, curved or flat? Do you know why the flight maps in airline magazines use curved lines to illustrate the different routes between airports?

Solid	Volume	Surface area	Notes
Cuboid 	$V = l \times w \times h$ where: l is the length, w is the width, h is the height.	$A = 2(lw + lh + wh)$	A cuboid is a prism with a rectangular base.
Right pyramid 	$V = \frac{1}{3}$ (area of base $\times$ vertical height)	A = area of base + area of all four triangles	The base can be either square or rectangular.
Cone 	$V = \frac{1}{3}\pi r^2 h$ where: r is the radius, h is the vertical height.	For the curved surface area, $A = \pi rl$ where: r is the radius, l is the slant height. For the total surface area, $A = \pi rl + \pi r^2$	A cone is a pyramid, but with a circular base.
Sphere 	$V = \frac{4}{3}\pi r^3$ where: r is the radius.	$A = 4\pi r^2$	A sphere is a solid in which all points on the surface are equidistant from the centre.
Hemisphere 	$V = \frac{2}{3}\pi r^3$	$A = 2\pi r^2$ (no base) $A = 2\pi r^2 + \pi r^2$ (with base)	A hemisphere is half a sphere; the base is circular.
Right prism 	V= area of cross-section $\times$ height or V = area of cross-section $\times$ length	A = 2(area of the base) + area of all the rectangular sides	Prisms are solids with the same area of cross-section for their entire length.
Cylinder 	$V = \pi r^2 h$ where: r is the radius, h is the height.	For the curved surface area, $A = 2\pi rh$ For the total surface area, $A = 2\pi rh + 2(\pi r^2)$	A cylinder is a prism with a circular base.

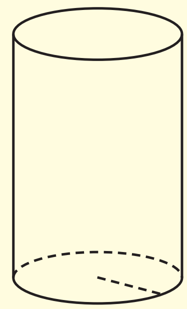
If you have problems visualising in three dimensions, there are many clear diagrams and descriptions on the Internet that you might find helpful. Try making models of the solids and rotating them, or look for objects in your environment that have the same shape.

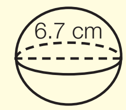
Your sketches do not have to be accurate – but do make sure that they are large enough and be prepared to erase them, start again, or adapt them in some way. You may not be right the first time!

Consider the shapes of buildings across the world; for instance, St Basil's Cathedral in Moscow, Hagia Sophia in Istanbul, Angkor Wat or the Taj Mahal. Architectural tradition, climate and the availability of building materials mean that they are very different and yet their basic building blocks are the classical solids first described in Ancient Greece. Look at pictures of modern architecture too and think about how the same concepts are interpreted. Is architectural design more universal now than it was when pyramids were being built in Central America and Egypt?

Worked example 16.7

Q. Three tennis balls are packed in a cylindrical container, just touching the sides, top and bottom of the container. The diameter of a tennis ball is 6.7 cm.





(a) Write down:

(i) the internal height of the container

(ii) the diameter of the container.

(b) Calculate the volume of the container, giving your answer to the nearest whole number.

The curved surface of the cylinder is made of plastic, and the top and bottom are made of metal. Calculate:

(c) the area of plastic used

(d) the area of metal used.

A. (a) (i) The internal height is: $h = 3 \times 6.7 = 20.1$ cm

(ii) $d = 6.7$ cm

The internal diameter of the container is the same as that of the tennis ball.

(b) $V = \pi r^2 h$

$V = \pi \times 3.35^2 \times 20.1 = 709\text{ cm}^3$

The radius is half the diameter.

(c) Area of plastic:

$2\pi rh = 2 \times \pi \times 3.35 \times 20.1 = 423\text{ cm}^2$

The formula for the curved surface area is $A = 2\pi rh$.

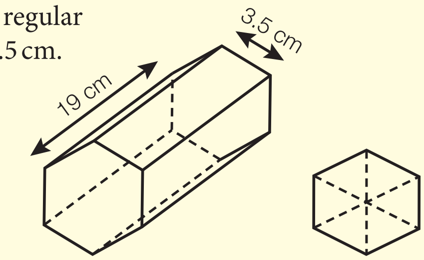
(d) Area of metal:

$2 \times \pi r^2 = 2 \times \pi \times 3.35^2 = 70.5\text{ cm}^2$

The top and bottom of the cylinder are circles, so use the formula for the area of a circle.

Worked example 16.8

Q. The base of a prism is a regular hexagon with sides of 3.5 cm. The length of the prism is 19 cm.

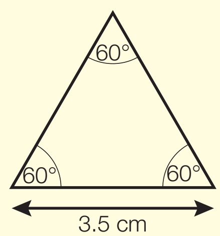


(a) Calculate the volume of the prism.

(b) The prism is to be made of flexible plastic and sold as a pencil case. Calculate the area of material needed for each case.

To calculate the volume, you must first calculate the area of cross-section of the prism. This may also be called the area of the base of the prism.

A. (a)



hint

Remember that an equilateral triangle has all three sides the same length and all three internal angles the same size.

In a regular polygon all the sides are the same length and all the angles are the same size. A regular hexagon consists of six **equilateral triangles**.

Use the formula $A = \frac{1}{2}ab\sin C$ to find the area of one triangle.

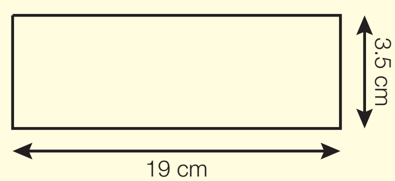
Area of triangle $= \frac{1}{2} \times 3.5 \times 3.5 \times \sin 60° = 5.304\ldots$

Area of hexagon $= 6 \times 5.304\ldots = 31.8\text{ cm}^2$

Volume of prism $= 19 \times 31.8 = 605\text{ cm}^3$

The surface area of the prism has two hexagonal ends and six rectangular sides.

(b)



Area of one rectangle $= 19 \times 3.5 = 66.5\text{ cm}^2$

Area of six rectangles $= 6 \times 66.5 = 399\text{ cm}^2$

Area of the two hexagonal ends $= 2 \times 31.8 = 63.6\text{ cm}^2$

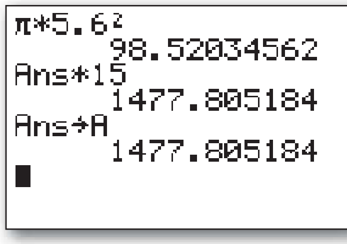
Total area of plastic $= 399 + 63.6 = 463\text{ cm}^2$

Using your GDC

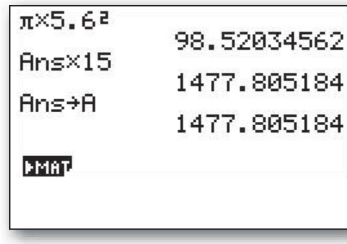
For many of the problems you meet, your GDC will give answers to several decimal places. Even though the IB asks for all answers to be given to three significant figures, do not round your answers too early. Problems like the ones encountered in this chapter will require several stages of calculation, and if you round the first answer and then use the rounded figures, you will be building up inaccuracies as you continue to work. It is better to use the **Ans/ANS** key or store your first solution in one of the memories of the calculator. (See '*22.2C The Ans/ANS key*' and '*22.2D Using the GDC memory*' on page 643 of the GDC chapter for a reminder of how to use these features if you need to.)

The screens below show the calculation to find the volume of a cylinder with a radius of 5.6 cm and a length of 15 cm.

TEXAS



CASIO



The area of the base is found; then the **Ans** key is used to substitute the area into the formula to find the volume.

Then the solution is stored in memory A.

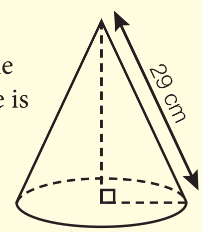
The final answer, to three significant figures, is $V = 1480\ \text{cm}^3$.

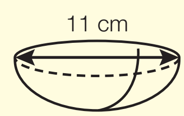
Worked example 16.9

Q. A lampshade is made from a cone of glass, with a glass hemisphere as its base. The slant height of the cone is 29 cm, and the diameter of the hemisphere is 11 cm. Calculate:

(a) the surface area of the lampshade

(b) the volume of the lampshade.





The diameter of the hemisphere is the same as the diameter of the cone.

A. (a) The radius of the cone is $\frac{1}{2} \times 11 = 5.5$ cm.

Cone: curved surface area $= \pi r l$

$A = \pi \times 5.5 \times 29$

Store 501.08... in memory A on your GDC.

$A = 501.08\ldots \text{cm}^2$

Hemisphere: surface area $= 2\pi r^2$

$A = 190.06\ldots \text{cm}^2$

Total area $= 691\ \text{cm}^3$

continued . . .

To find the volume of the cone, you need to know its height; use Pythagoras' theorem to do so.

TEXAS

```
A
        501.0840282
2*π*5.5²
        190.0663555
Ans+A
        691.1503838
█
```

CASIO

```
              501.0840282
A
              501.0840282
2×π×5.5²
              190.0663555
Ans+A
              691.1503838
▶MAT
```

(b) $h^2 = 29^2 - 5.5^2$

$h = \sqrt{810.75}$

$h = 28.5\text{ cm}$

Cone: volume $= \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3} \times \pi \times 5.5^2 \times 28.5$

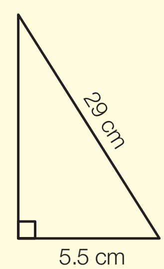
$V = 901.98\ldots\text{ cm}^3$

Hemisphere: volume $= \frac{2}{3}\pi r^3$

$V = \frac{2}{3} \times \pi \times 5.5^3$

$V = 348.45\ldots$

Total volume $= 1250\text{ cm}^3$



Exercise 16.3

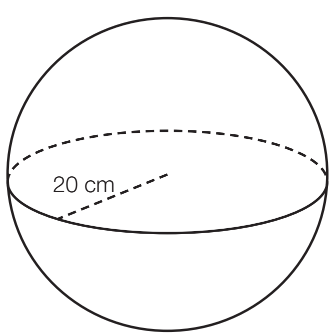
1. For each of the shapes in Exercise 16.1, questions 1–6, calculate:

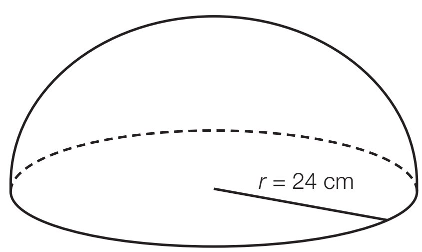
   (a) the total surface area (b) the volume.

   Give your answer to the nearest whole number. (Note: the solid in question 4 is an 'open' bowl so calculate the outside surface only.)

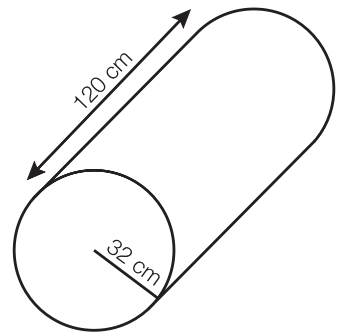
2. Calculate the volume of each of the following shapes, giving your answer to the nearest whole number:

   (a) a sphere with radius of 20 cm (b) a hemisphere with radius of 24 cm

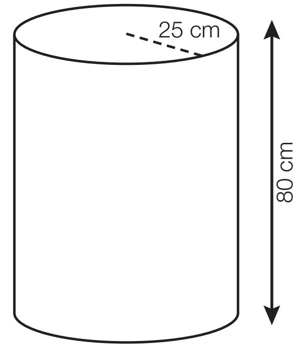




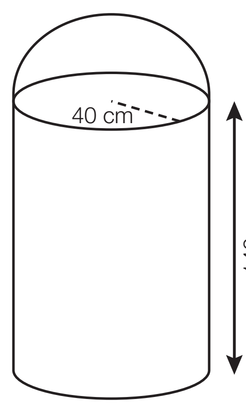
(c) a cylinder of radius 32 cm and length 120 cm



(d) a cylinder of radius 25 cm and height 80 cm



(e) a composite shape consisting of a hemisphere of radius 40 cm and a right cylinder of the same radius. The cylinder has a height of 140 cm.



3. Work out the total surface area of each of the shapes in question 2 above, giving your answer to the nearest whole number.

4. Felix is designing a container for detergent. The container is to have a volume of one litre (1000 cm^3). He is considering two options:

   (a) a spherical container of radius r cm

   (b) a cubical container of length x cm.

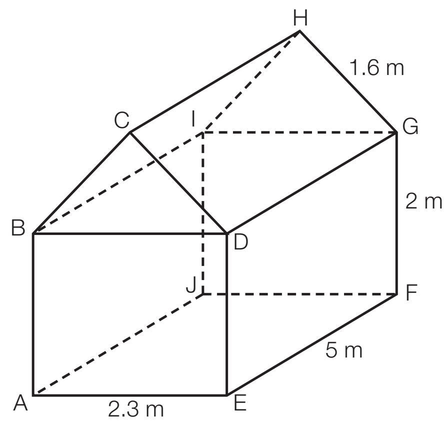
   Find the values of r and x.

5. A garden shed has a cross-section in the form of a pentagon ABCDE.

AB = ED = 2 m; BC = DC = 1.6 m; AE = BD = 2.3 m.

(a) Calculate the volume of the shed.

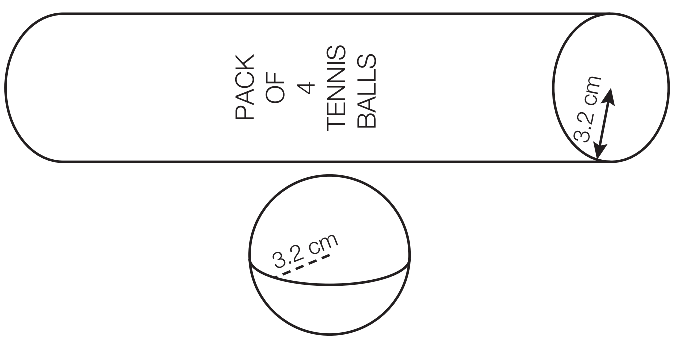
(b) Work out the total surface area of the roof of the shed BDGIHC.



6. A pack of tennis balls contains 4 balls fitted into a cylindrical container. The radius of each of the balls is 3.2 cm.

(a) Find the dimensions of the container if all 4 balls just fit into the container.

(b) Calculate the volume of unoccupied space in the container.



Summary

You should know:

- about the basic geometry of the following three-dimensional solids: cuboid, right prism, right pyramid, cylinder, sphere, hemisphere, cone and combinations of these solids
- how to calculate the distance between two points within a solid, such as the distance between two vertices, from a vertex to a midpoint, or from midpoint to midpoint
- how to calculate the size of an angle between two lines or between a line and a plane
- how to calculate the volume and surface area of the three-dimensional solids defined above.

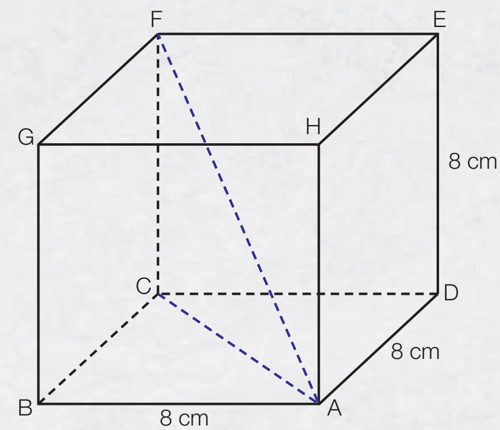
Mixed examination practice

Exam-style questions

1. A cube ABCDEFGH has sides of length 8 cm.

(a) Calculate the length of the diagonal AC.

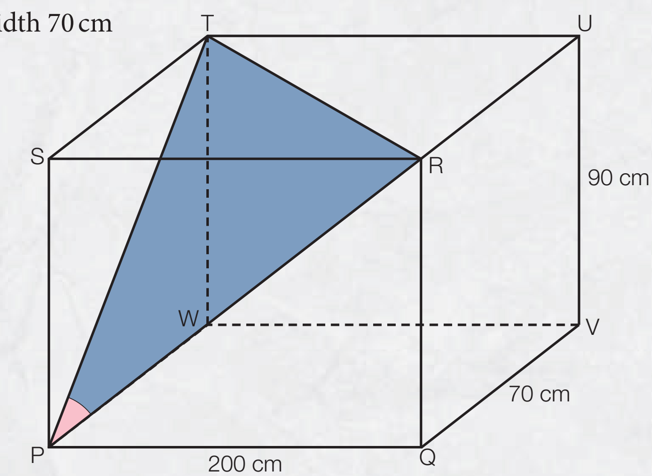
(b) Work out the length of the diagonal AF.



2. PQRSTUVW is a cuboid of length 200 cm, width 70 cm and height 90 cm.

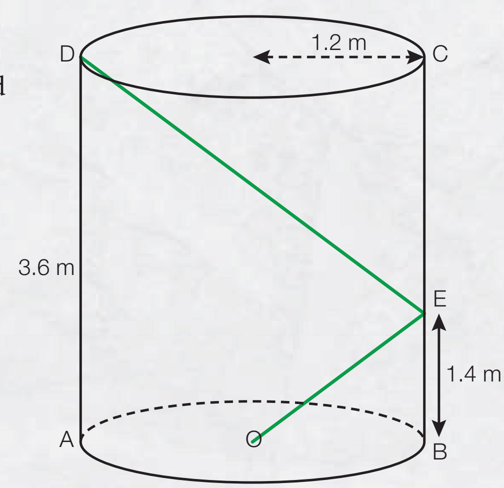
(a) Find the value of angle $R\hat{P}T$.

(b) Calculate angle $T\hat{Q}S$.



3. The diagram shows a cylindrical barrel of radius 1.2 metres and height 3.6 metres. A tube DEO is fixed into the cylinder. The tube consists of two straight parts OE and ED. O is at the centre of the base of the barrel and E lies along BC such that BE = 1.4 metres.

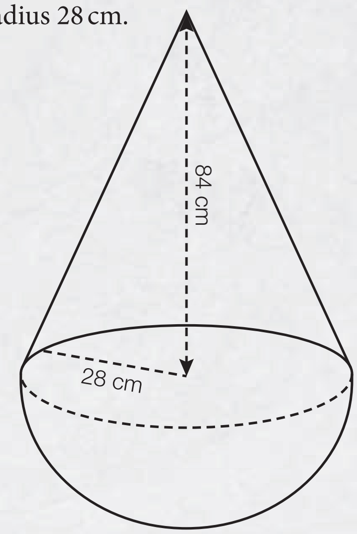
Calculate the total length of the tube.



4. A composite shape consists of a hemisphere and a **right cone**, both of radius 28 cm. The height of the cone is 84 cm.

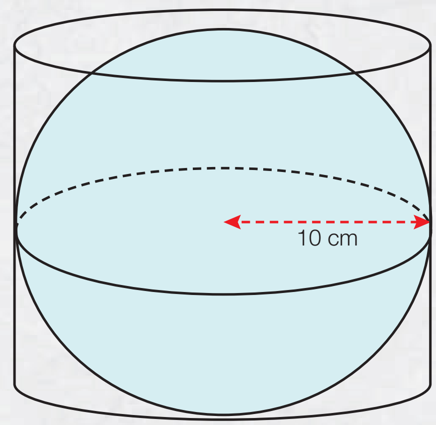
(a) Calculate the volume of the composite shape.

(b) Work out the total surface area of the shape.

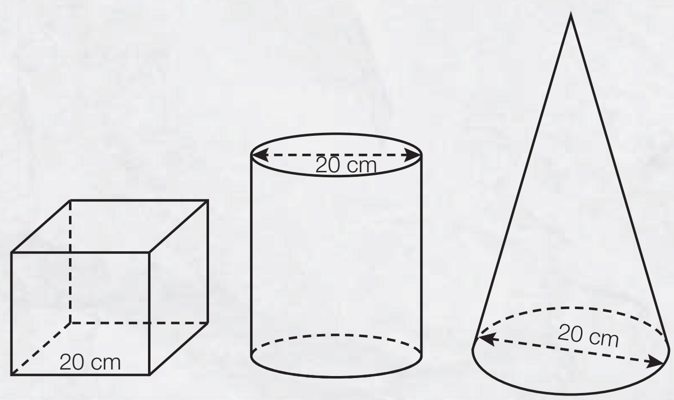


5. The following diagram shows a spherical ball of radius 10 cm, which just fits into a cylindrical container.

Calculate the volume of unoccupied space in the cylindrical container.



6. Each of the following three containers has a volume of 8000 cm^3.



Shape A Shape B Shape C

Shape A is a cube of side 20 cm.

Shape B is a cylinder of diameter 20 cm.

Shape C is a right cone of diameter 20 cm.

Work out the heights of shapes B and C.

7. Loretta has moulded a sphere of radius 12 cm out of clay. Later she remoulds four smaller spheres out of the original, bigger sphere.

   (a) Work out:

      (i) the volume of the original sphere

      (ii) the total surface area of the original sphere.

   (b) Find the volume of each of the smaller spheres.

   (c) Calculate the radius of the smaller sphere.

   (d) Would the total surface area of the four smaller spheres be the same as the surface area of the original sphere? Justify your answer.

Past paper questions

1. Jenny has a circular cylinder with a lid. The cylinder has height 39 cm and diameter 65 **mm**.

   (a) Calculate the volume of the cylinder in cm^3. Give your answer correct to **two** decimal places.

   [3 marks]

   The cylinder is used for storing tennis balls.

   Each ball has a **radius** of 3.25 cm.

   (b) Calculate how many balls Jenny can fit in the cylinder if it is filled to the top.

   [1 mark]

   (c) (i) Jenny fills the cylinder with the number of balls found in part (b) and puts the lid on. Calculate the volume of air inside the cylinder in the spaces between the tennis balls.

      (ii) Convert your answer to (c)(i) into cubic metres.

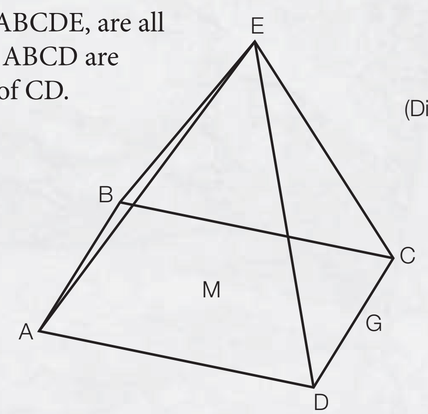
   [4 marks]

   [Total 8 marks]

   [May 2007, Paper 2, TZ0, Question 2(i)] (© *IB Organization 2007*)

2. The triangular faces of a square based pyramid, ABCDE, are all inclined at 70° to the base. The edges of the base ABCD are all 10 cm and M is the centre. G is the midpoint of CD.

(Diagram not to scale)



(a) Using the letters on the diagram draw a triangle showing the position of a 70° angle.

[1 mark]

(b) Show that the height of the pyramid is 13.7 cm, to 3 significant figures.

[2 marks]

(c) Calculate:

(i) the length of EG

(ii) the size of angle $D\hat{E}C$.

[4 marks]

(d) Find the total surface area of the pyramid.

[2 marks]

(e) Find the volume of the pyramid.

[2 marks]

[Total 11 marks]

[May 2008, Paper 2, TZ1, Question 4(ii)] (© *IB Organization 2008*)